# Making Triangulations 4-connected using Flips

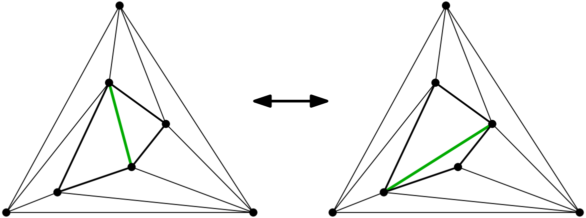
Prosenjit Bose, Dana Jansens, André van Renssen, Maria Saumell and Sander Verdonschot

Carleton University

August 8, 2011

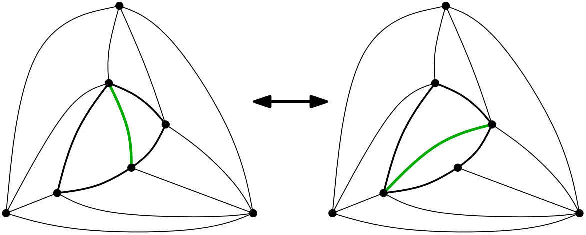
# Flips

- Replace one diagonal of a quadrilateral with the other



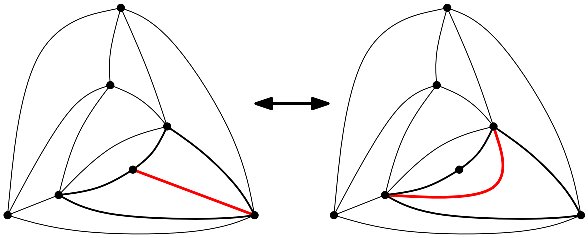
# Flips

- Replace one diagonal of a quadrilateral with the other



# Flips

- Replace one diagonal of a quadrilateral with the other



# Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip

# Flip Graph

- Vertex for each triangulation
- Edge if two triangulations differ by one flip
- Flip Distance: shortest path in flip graph

# Flip Graph

- Connected?

# Flip Graph

- Connected?
  - Yes - Wagner (1936)

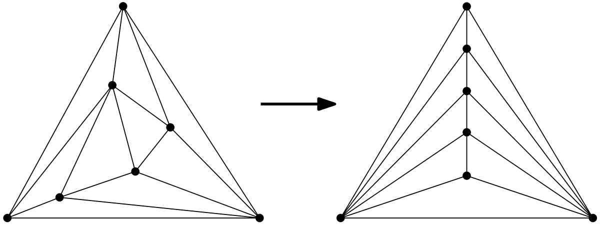
# Flip Graph

- Connected?
  - Yes - Wagner (1936)
- Diameter?
  - $O(n^2)$ - Wagner (1936)

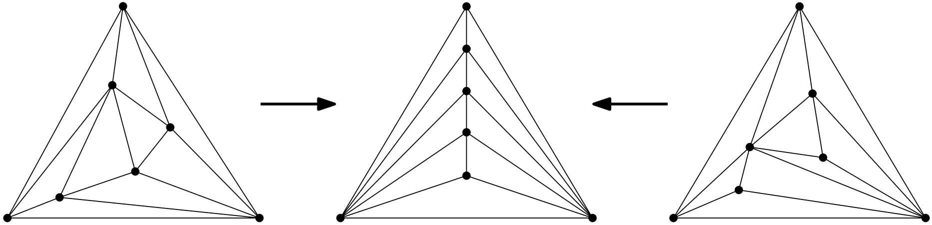
# Flip Graph

- Connected?
  - Yes - Wagner (1936)
- Diameter?
  - $O(n^2)$ - Wagner (1936)
  - $8n - 54$ - Komuro (1997)
  - $6n - 30$ - Mori *et al.* (2003)

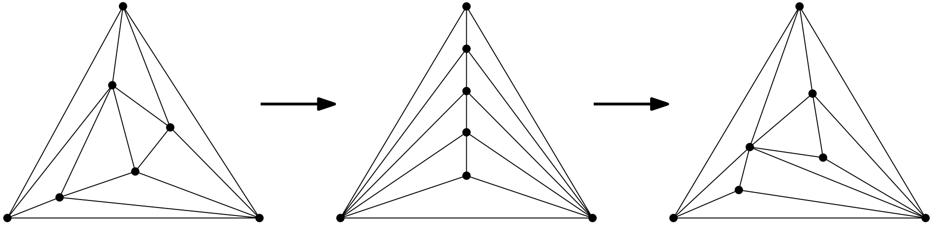
# Algorithm



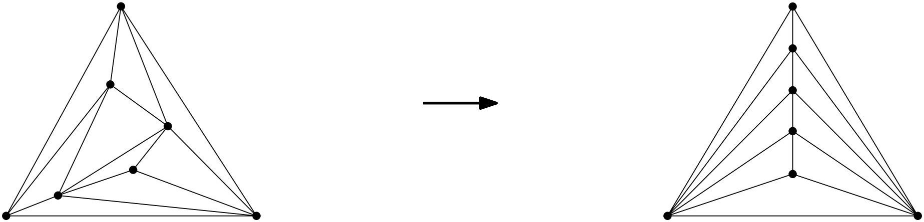
# Algorithm



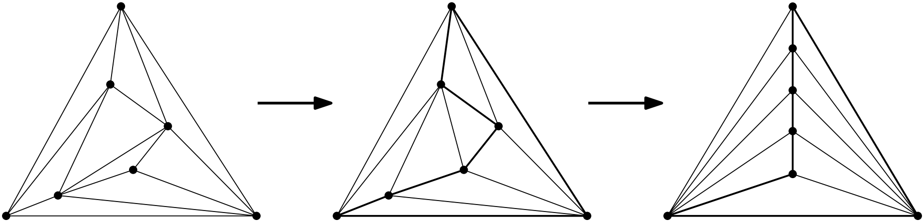
# Algorithm



# Algorithm Mori *et al.*

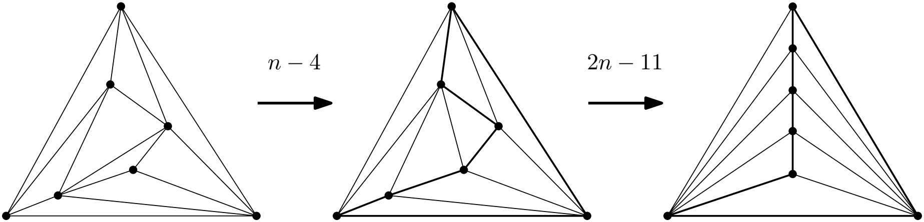


# Algorithm Mori *et al.*



4-connected ⇒ Hamiltonian

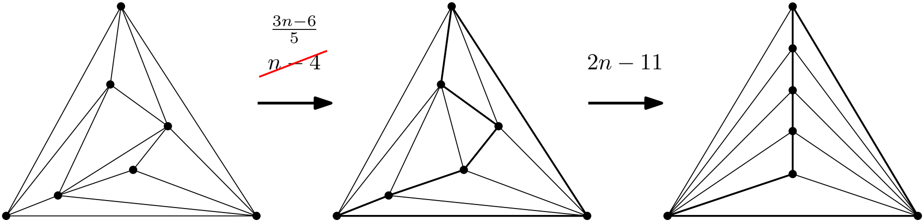
# Algorithm Mori *et al.*




4-connected $\Rightarrow$ Hamiltonian

Total: $6n - 30$

# Algorithm Mori *et al.*

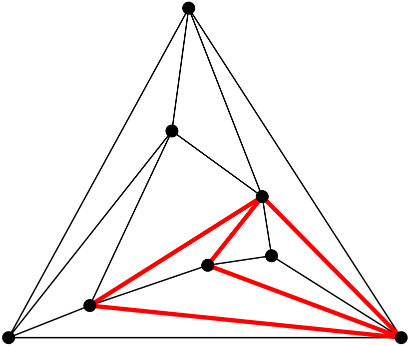



4-connected $\Rightarrow$ Hamiltonian

Total: ~~$6n-30$~~ $5.2n-24.4$

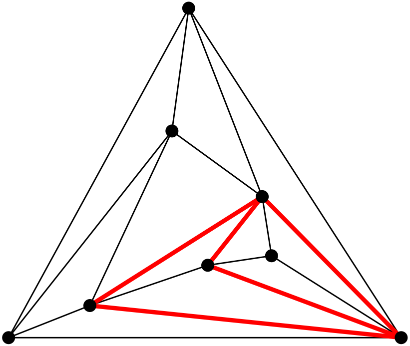
# Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph



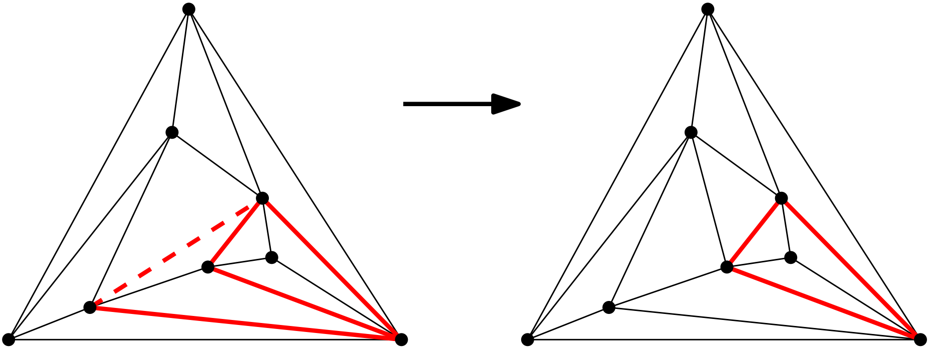
# Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected



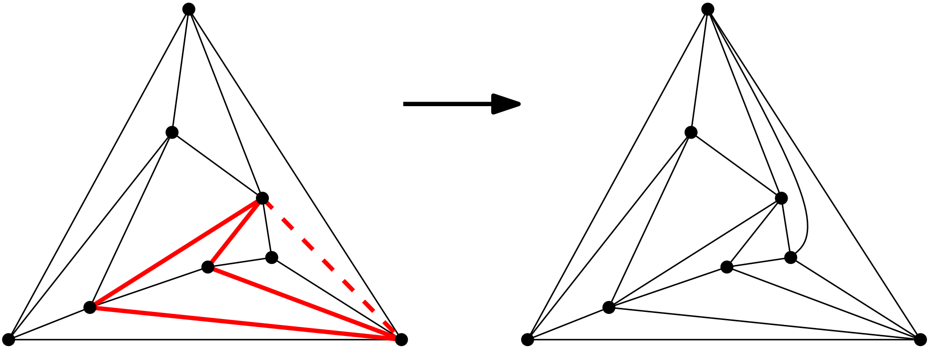
# Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it



# Making triangulations 4-connected

- *Separating triangle*: 3-cycle whose removal disconnects the graph
- No separating triangles $\iff$ 4-connected
- Flipping an edge of a separating triangle removes it
- Prefer shared edges



# Upper Bound

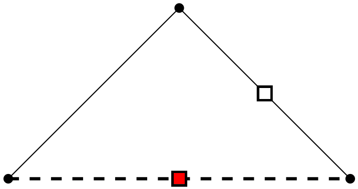
- To prove: $\#\text{flips} \leq (3n-6)/5$

# Upper Bound

- To prove: $\#\text{flips} \leq (3n-6)/5$
- Charging scheme:
  - Coin on every edge
  - Pay 5 coins per flip

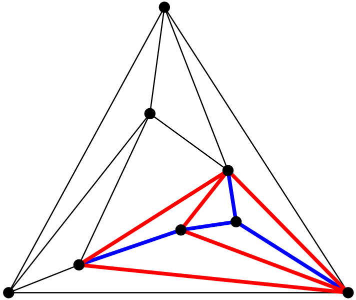
# Paying for flips

- Invariant: Every edge of a separating triangle has a coin
- Charge the flipped edge
- Charge all edges that aren't shared



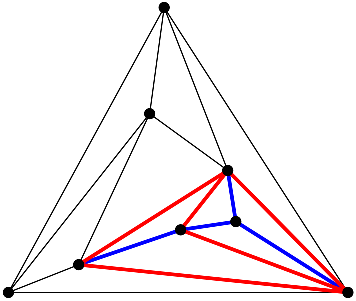
# Paying for flips

- *Free edge*: edge that is not part of any separating triangle



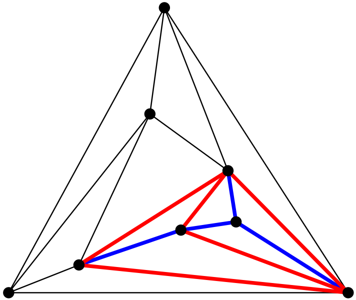
# Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- Every vertex of a separating triangle is incident to a free edge inside the triangle



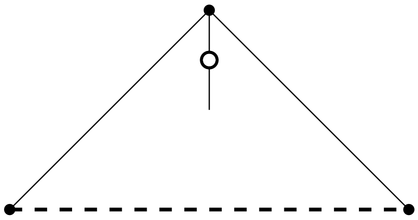
# Paying for flips

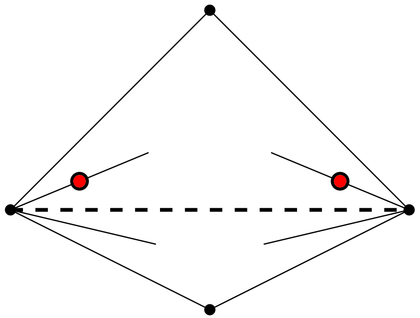
- *Free edge*: edge that is not part of any separating triangle
- *Invariant:* Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*



# Paying for flips

- *Free edge*: edge that is not part of any separating triangle
- *Invariant:* Every vertex of a separating triangle is incident to a free edge inside the triangle *that has a coin*
- Charge all free edges that aren't needed by other separating triangles





# Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles

# Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles
- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle

# Which edges to flip?

- A *deepest* separating triangle is contained in the maximum number of separating triangles
- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle

# Which edges to flip?

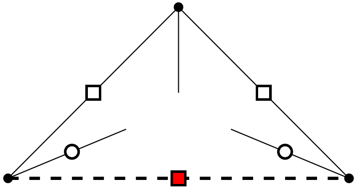
- A *deepest* separating triangle is contained in the maximum number of separating triangles
- Flip:
  - An arbitrary edge
  - Shared with other separating triangles
  - Not shared with a containing triangle

# Which edges to flip?

- Case 1: No shared edges

We can charge:

- The flipped edge
- An unshared triangle edge
- An unshared free edge
- A superfluous free edge

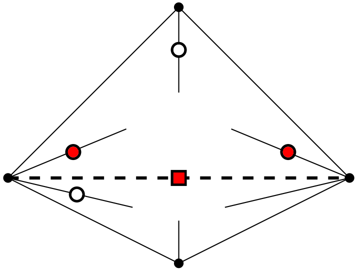


# Which edges to flip?

- Case 2: Shares edges with non-containing triangles

We can charge:

- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge

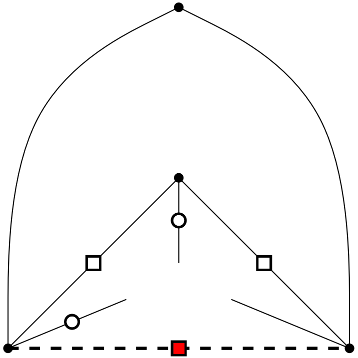


# Which edges to flip?

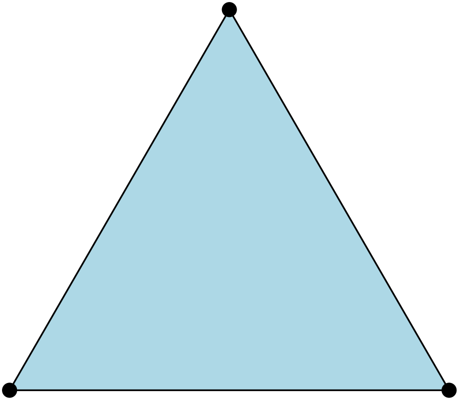
- Case 3: Shares one edge with containing triangle

We can charge:

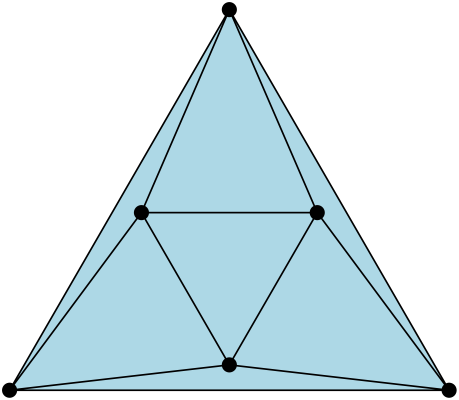
- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge



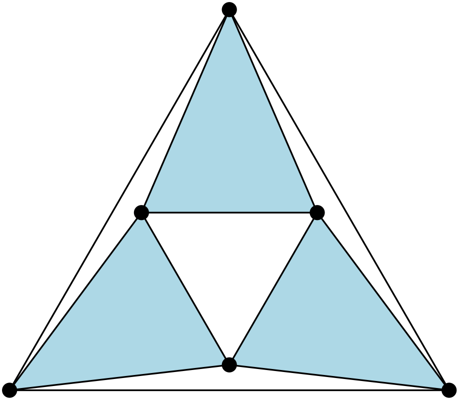
# Lower Bound



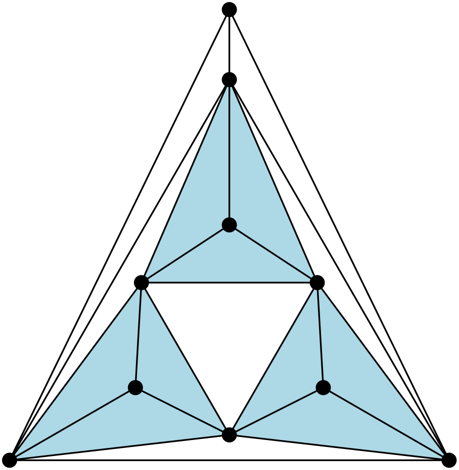
# Lower Bound



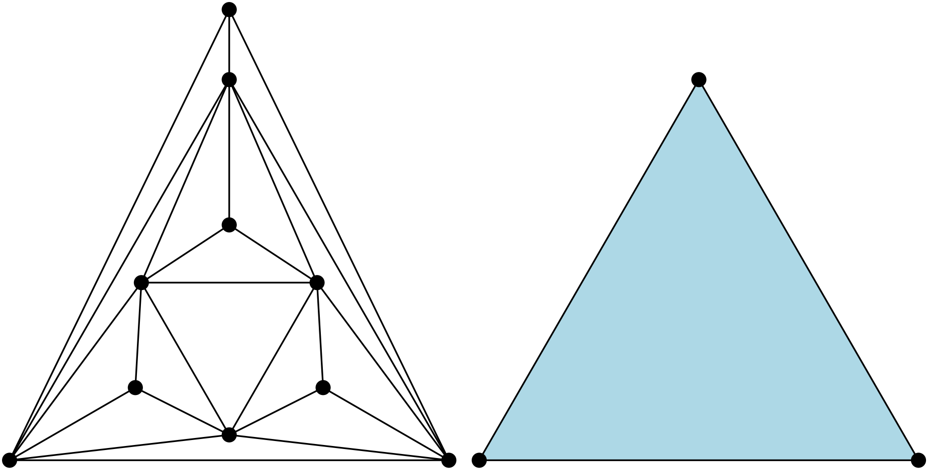
# Lower Bound



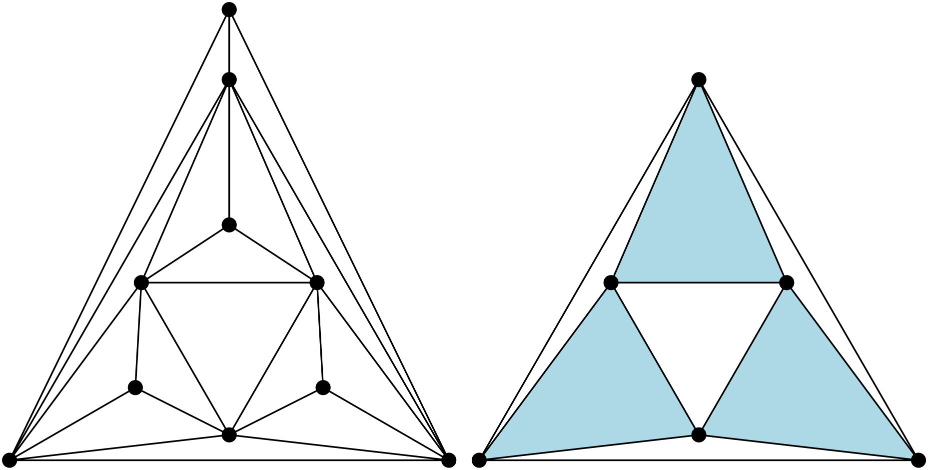
# Lower Bound



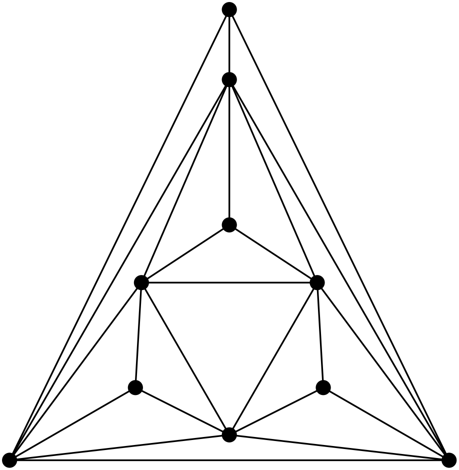
# Lower Bound

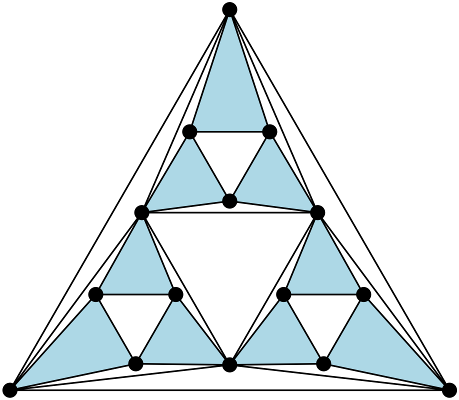


# Lower Bound

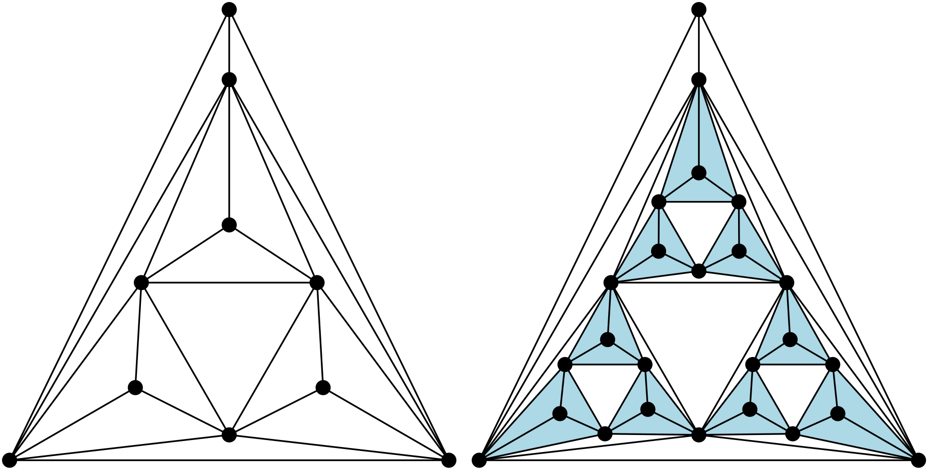


# Lower Bound



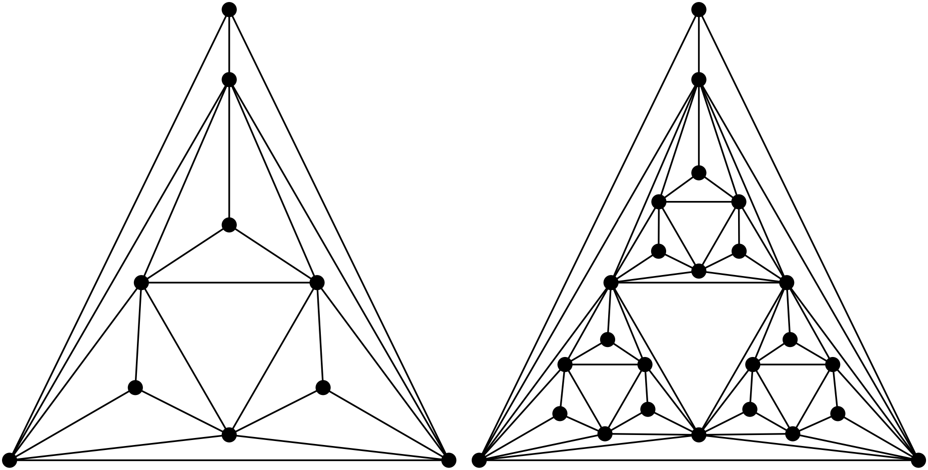


# Lower Bound



# Lower Bound

- $(3n - 10)/5$ edge-disjoint separating triangles



# Summary

- Any triangulation can be made 4-connected by $\left\lfloor \frac{3n-6}{5} \right\rfloor$ flips
- There are triangulations where this requires $\left\lceil \frac{3n-10}{5} \right\rceil$ flips

# Summary

- Any triangulation can be made 4-connected by $\left\lfloor \frac{3n-6}{5} \right\rfloor$ flips
- There are triangulations where this requires $\left\lceil \frac{3n-10}{5} \right\rceil$ flips
- A triangulation can be transformed into any other by $5.2n - 24.4$ flips

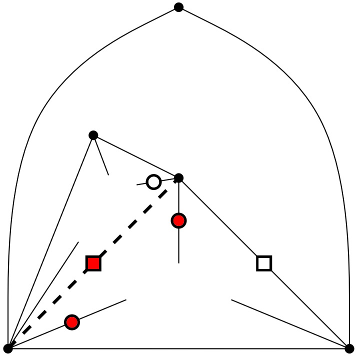
# The End

# Which edges to flip?

- Case 4: Shares an edge with containing triangle and one with non-containing triangle

We can charge:

- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge



# Which edges to flip?

- Case 5: Shares an edge with containing triangle and two with non-containing triangles

We can charge:

- ■ The flipped edge
- □ An unshared triangle edge
- ○ An unshared free edge
- ● A superfluous free edge