On the Spanning Ratio of Theta-Graphs*

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Abstract. We present improved upper bounds on the spanning ratio of a large family of θ-graphs. A θ-graph partitions the plane around each vertex into m disjoint cones, each having aperture $\theta = 2\pi/m$. We show that for any integer $k \geq 1$, θ-graphs with $4k + 4$ cones have spanning ratio at most $1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$. We also show that θ-graphs with $4k + 3$ and $4k + 5$ cones have spanning ratio at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. This is a significant improvement on all families of θ-graphs for which exact bounds are not known. For example, the spanning ratio of the θ-graph with 7 cones is decreased from at most 7.5625 to at most 3.5132. We also improve the upper bounds on the competitiveness of the θ-routing algorithm for these graphs to $1 + 2\sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$ on θ-graphs with $4k + 4$ cones and to $1 + 2\sin(\theta/2) \cdot \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$ on θ-graphs with $4k + 3$ and $4k + 5$ cones. For example, the routing ratio of the θ-graph with 7 cones is decreased from at most 7.5625 to at most 4.0490.

Keywords: computational geometry, spanners, θ-graphs, spanning ratio

1 Introduction

In a weighted graph G, let the distance $\delta_G(u, v)$ between two vertices u and v be the length of the shortest path between u and v in G. A subgraph H of G is a *t-spanner* of G if for all pairs of vertices u and v, $\delta_H(u, v) \leq t \cdot \delta_G(u, v), t \geq 1$. The *spanning ratio* of H is the smallest t for which H is a t-spanner. The graph G is referred to as the *underlying graph*. A routing strategy is said to be *c-competitive* if the length of the path returned by the routing strategy is not more than c times the length of the shortest path.

We consider the situation where the underlying graph G is a straightline embedding of K_n, the complete graph on a set of n points in the plane. The weight of each edge (u, v) is the Euclidean distance $|uv|$ between u and v. A spanner of such a graph is called a *geometric spanner*. We look at a specific type of geometric spanner: θ-graphs.

Introduced independently by Clarkson [5] and Keil [6], θ-graphs are constructed as follows (a more precise definition follows in the next section): for each vertex u, we partition the plane into m disjoint cones with apex u, each

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having aperture $\theta = 2\pi/m$. When m cones are used, we denote the resulting θ-graph as θ_m. The θ-graph is constructed by, for each cone with apex u, connecting u to the vertex v whose projection along the bisector of the cone is closest. Ruppert and Seidel [7] showed that the spanning ratio of these graphs is at most $1/(1-2\sin(\theta/2))$, when $\theta < \pi/3$, i.e. there are at least seven cones. This proof also showed that the *θ-routing* algorithm (defined in the next section) is $1/(1-2\sin(\theta/2))$-competitive on these graphs.

Bonichon *et al.* [1] showed that the θ_6-graph has spanning ratio 2. This was done by dividing the cones into two sets, positive and negative cones, such that each positive cone is adjacent to two negative cones and vice versa. It was shown that when edges are added only in the positive cones, in which case the graph is called the half-θ_6-graph, the resulting graph is equivalent to the TD-Delaunay triangulation (the Delaunay triangulation where the empty region is an equilateral triangle) whose spanning ratio is 2 as shown by Chew [4]. An alternative, inductive proof of the spanning ratio of the θ_6-graph was presented by Bose *et al.* [3] along with an optimal local competitive routing algorithm on the θ_6-graph. Recently, Bose *et al.* [2] generalized this inductive proof to show that the $\theta_{(4k+2)}$-graph has spanning ratio $1 + 2\sin(\theta/2)$, where k is an integer and at least 1. This spanning ratio is exact, i.e. there is a matching lower bound.

In this paper, we generalize the results from Bose *et al.* [2]. We look at the three remaining families of θ-graphs: the $\theta_{(4k+3)}$-graph, the $\theta_{(4k+4)}$-graph, and the $\theta_{(4k+5)}$-graph, where k is an integer and at least 1. We show that the $\theta_{(4k+4)}$-graph has a spanning ratio of at most $1+2\sin(\theta/2)/(\cos(\theta/2)-\sin(\theta/2))$. We also show that the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph have spanning ratio at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. We also improve the competitiveness of θ-routing on these graphs. The θ-routing algorithm is the standard routing algorithm on all θ-graphs having at least seven cones. For both the spanning ratio and the routing ratio, the best known bound was $1/(1 - 2\sin(\theta/2))$ by Ruppert and Seidel.

2 Preliminaries

Let a *cone* C be the region in the plane between two rays originating from the same point (referred to as the apex of the cone). For ease of exposition, we only consider point sets in general position: no two vertices lie on a line parallel to one of the rays that define the cones and no two vertices lie on a line perpendicular to the bisector of one of the cones.

When constructing a θ_m-graph, for each vertex u of K_n consider the rays originating from u with the angle between consecutive rays being $\theta = 2\pi/m$. Each pair of consecutive rays defines a cone. The cones are oriented such that the bisector of some cone coincides with the vertical halfline through u that lies above u. Let this cone be C_0 of u and number the cones in clockwise order around u. The cones around the other vertices have the same orientation as the ones around u. We write C_i^u to indicate the i-th cone of a vertex u.

The θ_m-graph is constructed as follows: for each cone C of each vertex u, add an edge from u to the closest vertex in that cone, where distance is measured along the bisector of the cone. More formally, we add an edge between two vertices u and v if $v \in C$ and for all vertices $w \in C$ $(v \neq w)$, $|uv'| \leq |uw'|$, where v' and w' denote the orthogonal projection of v and w on the bisector of C. Note that our general position assumption implies that each vertex adds at most one edge per cone to the graph.

Given a vertex w in cone C of vertex u, we define the *canonical triangle* T_{uw} as the triangle defined by the borders of C and the line through w perpendicular to the bisector of C. We use m to denote the midpoint of the side of T_{uw} opposite u and α to denote the unsigned angle between uw and um (see Figure 1). Note that for any pair of vertices u and w, there exist two canonical triangles: T_{uw} and T_{wu}.

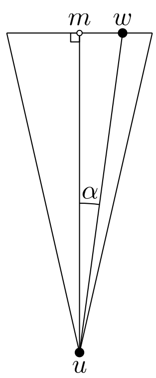

Fig. 1. The canonical triangle T_{uw}

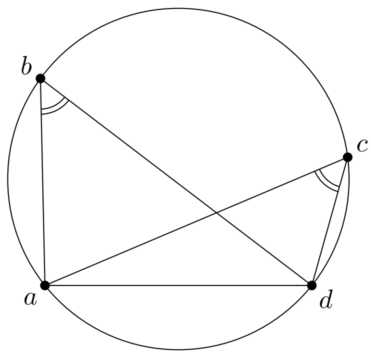

Fig. 2. Four points a, b, c, d on a circle

Using the structure of the θ_m-graph, *θ-routing* is defined as follows. Let t be the destination of the routing algorithm and let u be the current vertex. If there exists a direct edge to t, follow this edge. Otherwise, follow the edge to the closest vertex in T_{ut}.

Next, we prove a few geometric lemmas that will prove useful when bounding the spanning ratios of the graphs.

Lemma 1. *Let a, b, c, and d be four points on a circle such that $\angle cad \leq \angle bad \leq \angle adc$. It holds that $|ac| + |cd| \leq |ab| + |bd|$ and $|cd| \leq |bd|$.*

Proof. Since b and c lie on the same circle and $\angle abd$ and $\angle acd$ are the angle opposite to the same chord ad, the inscribed angle theorem implies that $\angle abd = \angle acd$ (see Figure 2). First, we show that $|ac| + |cd| \leq |ab| + |bd|$.

We look at the function $\sin\alpha + \sin(\pi - \gamma - \alpha)$. Using elementary calculus, it can be shown that this function is maximal at $\alpha = (\pi - \gamma)/2$ and strictly unimodal for $\alpha \in (0, \pi - \gamma)$. Next, we note that $|ac| + |cd| \leq |ab| + |bd|$ can be rewritten as

$2 \cdot r \cdot (\sin \angle adc + \sin \angle cad) \leq 2 \cdot r \cdot (\sin \angle adb + \sin \angle bad)$, where r is the radius of the circle. Since we can express $\angle adc$ and $\angle adb$ as $\pi - \angle acd - \angle cad$ and $\pi - \angle abd - \angle bad$, both sides of the inequality have the form $\sin \alpha + \sin(\pi - \gamma - \alpha)$. Hence, since $\angle cad \leq \angle bad \leq \pi - \angle acd - \angle cad = \angle adc$, we have that $|ac| + |cd| \leq |ab| + |bd|$.

Next, we show that $|cd| \leq |bd|$. The law of sines gives us that

$$\frac{|bd|}{\sin \angle bad} = \frac{|ad|}{\sin \angle abd} = \frac{|ad|}{\sin \angle acd} = \frac{|cd|}{\sin \angle cad}.$$

Hence we need to show that $\sin \angle bad \geq \sin \angle cad$. Since $\angle cad \leq \angle bad < \pi$, this is the case, concluding the proof of the lemma. ☐

Lemma 2. *Let u, v and w be three vertices in the $\theta_{(4k+x)}$-graph, $x \in \{3, 4, 5\}$, such that $w \in C_0^u$ and $v \in T_{uw}$, to the left of uw. Let a be the intersection of the side of T_{uw} opposite u and the left boundary of C_0^v. Let C_i^v denote the cone of v that contains w and let c and d be the upper and lower corner of T_{vw}. If $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, then $\max\{|vc| + |cw|, |vd| + |dw|\} \leq |va| + |aw|$ and $\max\{|cw|, |dw|\} \leq |aw|$.*

Proof. This situation is illustrated in Figure 3. We perform case distinction on $\max\{|cw|, |dw|\}$. If $|cw| > |dw|$, we need to show that when $1 \leq i \leq k-1$, $|vc| + |cw| \leq |va| + |aw|$ and $|cw| \leq |aw|$. Since angles $\angle vaw$ and $\angle vcw$ are both angles between the boundary of a cone and the line perpendicular to its bisector, $\angle vaw = \angle vcw$. Thus, c lies on the circle through a, v, and w. Therefore, if we can show that $\angle cvw \leq \angle avw \leq \angle vwc$, Lemma 1 proves the first half of the lemma.

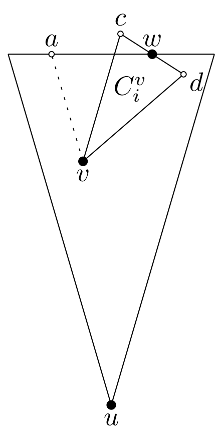

Fig. 3. The situation where we apply Lemma 1

We show $\angle cvw \leq \angle avw \leq \angle vwc$ in two steps. Since $w \in C_i^v$ and $i \geq 1$, we have that $\angle avc = i \cdot \theta \geq \theta$. Hence, since $\angle avw = \angle avc + \angle cvw$, $\angle cvw \leq \angle avw$. It remains to show that $\angle avw \leq \angle vwc$. We note that $\angle avw \leq (i+1) \cdot \theta$ and

$(\pi - \theta)/2 \leq \angle vwc$. Using that $\theta = 2\pi/(4k + x)$ and $x \in \{3, 4, 5\}$, we compute the maximum value of i for which $\angle avw \leq \angle vwc$:

$$\begin{aligned}
\angle avw &\leq \angle vwc \\
(i+1) \cdot \theta &\leq \frac{\pi - \theta}{2} \\
i &\leq \frac{\pi}{2\theta} - \frac{3}{2} \\
i &\leq \frac{\pi \cdot (4k + x)}{4\pi} - \frac{3}{2} \\
i &\leq k + \frac{x}{4} - \frac{3}{2} \\
i &\leq k - 1
\end{aligned}$$

Hence, $\angle avw \leq \angle vwc$ when $i \leq k - 1$.

If $|cw| \leq |dw|$, we need to show that when $1 \leq i \leq k$, $|vd| + |dw| \leq |va| + |aw|$ and $|dw| \leq |aw|$. Since angles $\angle vaw$ and $\angle vdw$ are both angles between the boundary of a cone and the line perpendicular to its bisector, $\angle vaw = \angle vcw$. Thus, when we reflect d around vw, the resulting point d' lies on the circle through a, v, and w. Therefore, if we can show that $\angle d'vw \leq \angle avw \leq \angle vwd'$, Lemma 1 proves the second half of the lemma.

We show $\angle d'vw \leq \angle avw \leq \angle vwd'$ in two steps. Since $w \in C_i^v$ and $i \geq 1$, we have that $\angle avw \geq \angle avc = i \cdot \theta \geq \theta$. Hence, since $\angle d'vw \leq \theta$, $\angle d'vw \leq \angle avw$. It remains to show that $\angle avw \leq \angle vwd'$. We note that $\angle vwd' = \angle dwv = \pi - (\pi - \theta)/2 - \angle dvw$ and $\angle avw = \angle avd - \angle dvw = (i+1) \cdot \theta - \angle dvw$. Using that $\theta = 2\pi/(4k+x)$ and $x \in \{3, 4, 5\}$, we compute the maximum value of i for which $\angle avw \leq \angle vwd'$:

$$\begin{aligned}
\angle avw &\leq \angle vwd' \\
(i+1) \cdot \theta - \angle dvw &\leq \frac{\pi + \theta}{2} - \angle dvw \\
i &\leq \frac{\pi}{2\theta} - \frac{1}{2} \\
i &\leq \frac{\pi \cdot (4k + x)}{4\pi} - \frac{1}{2} \\
i &\leq k + \frac{x}{4} - \frac{1}{2} \\
i &\leq k
\end{aligned}$$

Hence, $\angle avw \leq \angle vwd'$ when $i \leq k$. □

Lemma 3. *Let u, v and w be three vertices in the $\theta_{(4k+x)}$-graph, such that $w \in C_0^u$ and $v \in T_{uw}$, to the left of uw. Let a be the intersection of the side of T_{uw} opposite u and the line through v parallel to the left boundary of T_{uw}. Let y and z be the corners of T_{vw} opposite to u. Let $\beta = \angle awv$ and let γ be the angle between vw and the bisector of T_{vw}. Let $\boldsymbol{c}$ be a constant at least 1. If*

$$\boldsymbol{c} \geq \frac{\cos\gamma - \sin\beta}{\cos\left(\frac{\theta}{2} - \beta\right) - \sin\left(\frac{\theta}{2} + \gamma\right)},$$

then

$$|vp| + \boldsymbol{c} \cdot |pw| \leq |va| + \boldsymbol{c} \cdot |aw|,$$

where p *is* y *if* $|yw| \geq |zw|$ *and* z *if* $|yw| < |zw|$.

Proof. Using that the angle between the bisector of a cone and its boundary is $\theta/2$, we first express the four line segments in terms of β and γ (see Figure 4):

$$\begin{aligned}
|vp| &= |vw| \cdot \cos\gamma / \cos(\theta/2) \\
|pw| &= |vw| \cdot (\sin\gamma + \cos\gamma \cdot \tan(\theta/2)) \\
|va| &= |vw| \cdot \sin\beta / \cos(\theta/2) \\
|aw| &= |vw| \cdot (\cos\beta + \sin\beta \cdot \tan(\theta/2))
\end{aligned}$$

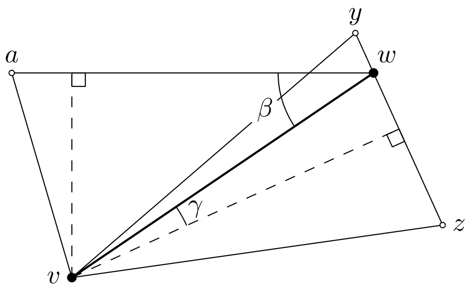

Fig. 4. Finding a constant $\boldsymbol{c}$ such that $|vz| + \boldsymbol{c} \cdot |zw| \leq |va| + \boldsymbol{c} \cdot |aw|$

To compute for which values of $\boldsymbol{c}$ the inequality $|vp| + \boldsymbol{c} \cdot |pw| \leq |va| + \boldsymbol{c} \cdot |aw|$ holds, we first multiply both sides by $\cos(\theta/2)/|vw|$ and rewrite as follows:

$$\begin{aligned}
|vp| + |pw| \cdot \boldsymbol{c} &= \cos\gamma + \boldsymbol{c} \cdot (\sin\gamma \cdot \cos(\theta/2) + \cos\gamma \cdot \sin(\theta/2)) \\
&= \cos\gamma + \boldsymbol{c} \cdot \sin(\theta/2 + \gamma)
\end{aligned}$$

$$\begin{aligned}
|va| + |aw| \cdot \boldsymbol{c} &= \sin\beta + \boldsymbol{c} \cdot (\cos\beta \cdot \cos(\theta/2) + \sin\beta \cdot \sin(\theta/2)) \\
&= \sin\beta + \boldsymbol{c} \cdot \cos(\theta/2 - \beta)
\end{aligned}$$

We can now calculate for which values of $\boldsymbol{c}$ the inequality holds:

$$\begin{aligned}
\cos\gamma + \boldsymbol{c} \cdot \sin(\theta/2 + \gamma) &\leq \sin\beta + \boldsymbol{c} \cdot \cos(\theta/2 - \beta) \\
\cos\gamma - \sin\beta &\leq \boldsymbol{c} \cdot (\cos(\theta/2 - \beta) - \sin(\theta/2 + \gamma)) \\
\boldsymbol{c} &\geq \frac{\cos\gamma - \sin\beta}{\cos(\theta/2 - \beta) - \sin(\theta/2 + \gamma)}
\end{aligned}$$

□

3 Generic Spanning Proof

Using the lemmas from the previous section, we provide a generic spanning proof for the three families of θ-graphs. After providing this proof, we fill in the blanks for the individual families.

Theorem 1. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path in the $\theta_{(4k+x)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha \right) \cdot \boldsymbol{c} \right) \cdot |uw|,$$

where $\boldsymbol{c} \geq 1$ is a constant that depends on $x \in \{3, 4, 5\}$. For the $\theta_{(4k+4)}$-graph, $\boldsymbol{c}$ equals $1/(\cos(\theta/2) - \sin(\theta/2))$ and for the $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph, $\boldsymbol{c}$ equals $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Proof. We prove the theorem by induction on the area of T_{uw} (formally, induction on the rank, when ordered by area, of the canonical triangles for all pairs of vertices). Let a and b be the upper left and right corners of T_{uw}, let y and z be the left and right intersections of T_{uw} and the lines through w parallel to the boundaries of the cone of u that contains w, and let r and s be the intersections of T_{uw} and the lines through w such that Lemma 2 can be applied to triangles yrw and zsw (see Figure 5). This region depends on x. The precise locations of r and s are specified in the spanning proofs of the three families.

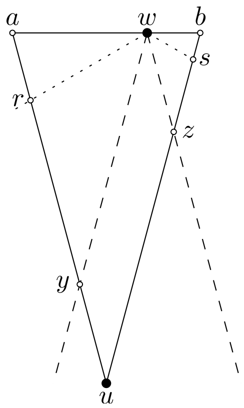

Fig. 5. The canonical triangle T_{uw} with a, b, r, s, y, and z being the various intersections with its sides, in this case for the θ_{12}-graph

Our inductive hypothesis is the following, where $\delta(u, w)$ denotes the length of the shortest path from u to w in the $\theta_{(4k+x)}$-graph: $\delta(u, w) \leq \max\{|ua| + |aw| \cdot \boldsymbol{c}, |ub| + |bw| \cdot \boldsymbol{c}\}$.

We first show that this induction hypothesis implies the theorem. Basic trigonometry gives us the following equalities: $|um| = |uw| \cdot \cos\alpha$, $|mw| = |uw| \cdot \sin\alpha$, $|am| = |bm| = |uw| \cdot \cos\alpha \cdot \tan(\theta/2)$, and $|ua| = |ub| = |uw| \cdot \cos\alpha / \cos(\theta/2)$. Thus the induction hypothesis gives that $\delta(u, w)$ is at most $|ua| + (|am| + |mw|) \cdot \boldsymbol{c} = |uw| \cdot (\cos\alpha/\cos(\theta/2) + (\cos\alpha \cdot \tan(\theta/2) + \sin\alpha) \cdot \boldsymbol{c})$.

Base case: T_{uw} has rank 1. Since the triangle is a smallest triangle, w is the closest vertex to u in that cone. Hence the edge (u, w) is part of the $\theta_{(4k+x)}$-graph, and $\delta(u, w) = |uw|$. From the triangle inequality and the fact that $\boldsymbol{c} \geq 1$, we have $|uw| \leq \max\{|ua| + |aw| \cdot \boldsymbol{c}, |ub| + |bw| \cdot \boldsymbol{c}\}$, so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices with canonical triangles of rank up to j. Let T_{uw} be a canonical triangle of rank $j + 1$.

If (u, w) is an edge in the $\theta_{(4k+x)}$-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between u and w, let v be the vertex closest to u in T_{uw}, and let a' and b' be the upper left and right corners of T_{uv} (see Figure 6). By definition, $\delta(u, w) \leq |uv| + \delta(v, w)$, and by the triangle inequality, $|uv| \leq \min\{|ua'| + |a'v|, |ub'| + |b'v|\}$.

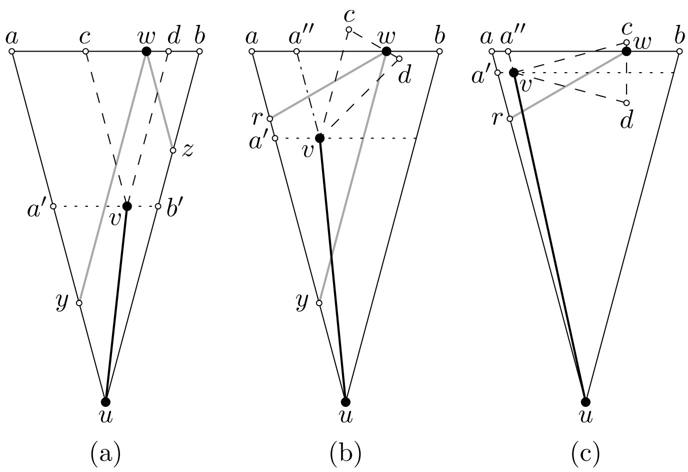

Fig. 6. The three cases based on the location of v, in this case for the θ_{12}-graph

We perform a case analysis based on the location of v: (a) v lies in $uywz$, (b) v lies in yrw, (c) v lies in raw, (d) v lies in zsw, (e) v lies in sbw (see Figure 5). Since Case (d) is analogous to Case (b) and Case (e) is analogous to Case (c), we discuss only the first three cases.

Case (a): Vertex v lies in $uywz$. Let c and d be the upper left and right corners of T_{vw} (see Figure 6a). Since T_{vw} has smaller area than T_{uw}, we apply the inductive hypothesis on T_{vw}.

Hence we have $\delta(v, w) \leq \max\{|vc| + |cw| \cdot \boldsymbol{c}, |vd| + |dw| \cdot \boldsymbol{c}\}$. Assume, without loss of generality, that the maximum of the left hand side is attained by its second argument $|vc| + |cw| \cdot \boldsymbol{c}$ (the other case is analogous). Since vertices v, c, a, and a' form a parallelogram and $\boldsymbol{c} \geq 1$, we have that:

$$\begin{aligned}\delta(u, w) &\leq |uv| + \delta(v, w)\\ &\leq |ua'| + |a'v| + |vc| + |cw| \cdot \boldsymbol{c}\\ &\leq |ua| + |aw| \cdot \boldsymbol{c}\\ &\leq \max\{|ua| + |aw| \cdot \boldsymbol{c}, |ub| + |bw| \cdot \boldsymbol{c}\},\end{aligned}$$

which proves the induction hypothesis.

Case (b): Since T_{vw} is smaller than T_{uw}, by induction we have $\delta(v, w) \leq \max\{|vc|+|cw|\cdot\boldsymbol{c}, |vd|+|dw|\cdot\boldsymbol{c}\}$ (see Figure 6b). Since by definition yrw is the triangle where we can apply Lemma 2, we get that $\max\{|vc| + |cw|, |vd| + |dw|\} \leq |va''| + |a''w|$ and $\max\{|cw|, |dw|\} \leq |a''w|$. Since $\boldsymbol{c} \geq 1$, this implies that $\max\{|vc| + |cw| \cdot \boldsymbol{c}, |vd| + |dw| \cdot \boldsymbol{c}\} \leq |va''| + |a''w| \cdot \boldsymbol{c}$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we have that $\delta(u, w) \leq |ua| + |aw| \cdot \boldsymbol{c}$, proving the induction hypothesis for T_{uw}.

Case (c) Vertex v lies in raw. Since T_{vw} is smaller than T_{uw}, we can apply induction on it. The precise application of the induction hypothesis varies for the three families of θ-graphs and, using Lemma 3, determines the value of $\boldsymbol{c}$. Hence, this case is discussed in the spanning proofs of the three families. □

4 The $\boldsymbol{\theta_{(4k+4)}}$-Graph

In this section, we give improved upper bounds on the spanning ratio of the $\theta_{(4k+4)}$-graph, for any integer $k \geq 1$.

Theorem 2. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path in the $\theta_{(4k+4)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right) \cdot |uw|.$$

Proof. We apply Theorem 1 using $\boldsymbol{c} = 1/(\cos(\theta/2) - \sin(\theta/2))$. First, we have to place r and s such that Lemma 2 can be applied to yrw and zsw. To this end, we pick r and s to be the intersections of T_{uw} and the lines through w such that $\angle awr = \angle bws = \theta$, i.e. the line through w such that $w \in C_i^v$ with either $1 \leq i \leq k$, or $i = k$ and $|cw| \leq |dw|$. Next, it remains to handle Case (c), where v lies in raw.

Case (c): Vertex v lies in raw. Let c and d be the upper and lower right corners of T_{vw} and let a'' be the intersection of aw and the line through v, parallel to ua (see Figure 6c). Let p be the intersection of the left boundary of T_{uw} and the line through w such that $\angle awp = \theta/2$. Let β be $\angle a''wv$ and let γ

be the angle between vw and the bisector of T_{vw}. We split this case into two subcases, depending on the location of v: (i) v lies below pw, (ii) v lies above pw. Since T_{vw} is smaller than T_{uw}, the induction hypothesis gives a bound on $\delta(v, w)$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we need to show that $\delta(v, w) \leq |va''| + |a''w| \cdot \boldsymbol{c}$ for both cases in order to complete the proof.

Case (i): When v lies below pw, w lies below the bisector of T_{vw} and the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vc| + |cw| \cdot \boldsymbol{c}$. We note that $\gamma = \theta - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\theta - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/2 - \beta))$. As this function is decreasing in β for $\theta/2 \leq \beta \leq 3\theta/4$, it is maximized when β equals $\theta/2$. Hence $\boldsymbol{c}$ needs to be at least $(\cos(\theta/2) - \sin(\theta/2))/(1 - \sin\theta)$, which can be rewritten to $1/(\cos(\theta/2) - \sin(\theta/2))$.

Case (ii): When v lies above pw, w lies above the bisector of T_{vw} and the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |wd| + |dv| \cdot \boldsymbol{c}$. We note that $\gamma = \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos\beta - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(\theta/2 + \beta))$. As this function is decreasing in β for $0 \leq \beta \leq \theta/2$, it is maximized when β equals 0. Hence $\boldsymbol{c}$ needs to be at least $1/(\cos(\theta/2) - \sin(\theta/2))$. □

Since $\cos\alpha/\cos(\theta/2) + (\cos\alpha \cdot \tan(\theta/2) + \sin\alpha)/(\cos(\theta/2) - \sin(\theta/2))$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq \pi/4$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

Corollary 1. *The $\theta_{(4k+4)}$-graph is a $\left(1 + \frac{2\cdot\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right)$-spanner of K_n.*

Furthermore, we observe that the proof of Theorem 2 follows the same path as the θ-routing algorithm follows: if the direct edge to the destination is part of the graph, it follows this edge, and if it is not, it follows the edge to the closest vertex in the cone that contains the destination.

Corollary 2. *The θ-routing algorithm is $\left(1 + \frac{2\cdot\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}\right)$-competitive on the $\theta_{(4k+4)}$-graph.*

5 The $\boldsymbol{\theta_{(4k+3)}}$-Graph and the $\boldsymbol{\theta_{(4k+5)}}$-Graph

In this section, we give improved upper bounds on the spanning ratio of the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph, for any integer $k \geq 1$.

Theorem 3. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path in the $\theta_{(4k+3)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \right) \cdot |uw|.$$

Proof. We apply Theorem 1 using $\boldsymbol{c} = \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. First, we have to place r and s such that Lemma 2 can be applied to yrw and zsw. To this end, we pick r and s to be the intersections of T_{uw} and the lines through w such that $\angle awr = \angle bws = 3\theta/4$, i.e. the line through w such that $w \in C_i^v$ with either $1 \leq i \leq k$, or $i = k$ and $|cw| \leq |dw|$. Next, it remains to handle Case (c), where v lies in raw.

Case (c): Vertex v lies in raw. Let c and d be the upper and lower right corners of T_{vw} and let a'' be the intersection of aw and the line through v, parallel to ua (see Figure 6c). Let p be the intersection of the left boundary of T_{uw} and the line through w such that $\angle awp = \theta/4$. Let β be $\angle a''wv$ and let γ be the angle between vw and the bisector of T_{vw}. We split this case into two subcases, depending on the location of v: (i) v lies below pw, (ii) v lies above pw. Since T_{vw} is smaller than T_{uw}, the induction hypothesis gives a bound on $\delta(v,w)$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we need to show that $\delta(v,w) \leq |va''| + |a''w| \cdot \boldsymbol{c}$ for both cases in order to complete the proof.

Case (i): When v lies below pw, w lies below the bisector of T_{vw} and the induction hypothesis for T_{vw} gives $\delta(v,w) \leq |vc| + |cw| \cdot \boldsymbol{c}$. We note that $\gamma = 3\theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(3\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(5\theta/4 - \beta))$. As this function is decreasing in β for $\theta/4 \leq \beta \leq 3\theta/4$, it is maximized when β equals $\theta/4$. Hence $\boldsymbol{c}$ needs to be at least $\boldsymbol{c} \geq (\cos(\theta/2) - \sin(\theta/4))/(\cos(\theta/4) - \sin\theta)$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (ii): When v lies above pw, w lies above the bisector of T_{vw} and the induction hypothesis for T_{vw} gives $\delta(v,w) \leq |wd| + |dv| \cdot \boldsymbol{c}$. We note that $\gamma = \theta/4 + \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\theta/4 + \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 + \beta))$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. □

A similar proof gives the same result for the $\theta_{(4k+5)}$-graph. Due to space constraints, this proof can be found in the appendix.

Theorem 4. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path in the $\theta_{(4k+5)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \right) \cdot |uw|.$$

When looking at two vertices u and w in the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph, we notice that when the angle between uw and the bisector of T_{uw} is α, the angle between wu and the bisector of T_{wu} is $\theta/2 - \alpha$. Hence the worst case spanning ratio becomes the minimum of the spanning ratio when looking at T_{uw} and the spanning ratio when looking at T_{wu}.

Theorem 5. *The $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph are $\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)}$-spanners of K_n.*

Proof. The spanning ratio of the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph is at most:

$$\min\left\{\begin{array}{l}\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)}+\frac{\left(\cos\alpha\cdot\tan\left(\frac{\theta}{2}\right)+\sin\alpha\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)},\\ \frac{\cos\left(\frac{\theta}{2}-\alpha\right)}{\cos\left(\frac{\theta}{2}\right)}+\frac{\left(\cos\left(\frac{\theta}{2}-\alpha\right)\cdot\tan\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}-\alpha\right)\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)}\end{array}\right\}$$

Since $\cos\alpha/\cos(\theta/2)+(\cos\alpha\cdot\tan(\theta/2)+\sin\alpha)\cdot\boldsymbol{c}$ is increasing for $\alpha\in[0,\theta/2]$, for $\theta\leq 2\pi/7$, the minimum of these two functions is maximized when the two functions are equal, i.e. when $\alpha=\theta/4$. Thus the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph has spanning ratio at most:

$$\frac{\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)}+\frac{\left(\cos\left(\frac{\theta}{4}\right)\cdot\tan\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{4}\right)\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)}=\frac{\cos\left(\frac{\theta}{4}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\cdot\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)\right)}$$

□

Furthermore, we observe that the proofs of Theorem 3 and Theorem 4 follow the same path as the θ-routing algorithm follows.

Theorem 6. *The θ-routing algorithm is $1+\frac{2\cdot\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{4}\right)}$-competitive on the $\theta_{(4k+3)}$-graph and the $\theta_{(4k+5)}$-graph.*

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A Spanning Proof of the $\theta_{(4k+5)}$-Graph

Theorem 7. *Let u and w be two vertices in the plane. Let m be the midpoint of the side of T_{uw} opposite u and let α be the unsigned angle between uw and um. There exists a path in the $\theta_{(4k+5)}$-graph of length at most*

$$\left(\frac{\cos\alpha}{\cos\left(\frac{\theta}{2}\right)} + \frac{\left(\cos\alpha \cdot \tan\left(\frac{\theta}{2}\right) + \sin\alpha\right) \cdot \cos\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{4}\right)} \right) \cdot |uw|.$$

Proof. We apply Theorem 1 using $\boldsymbol{c} = \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. First, we have to place r and s such that Lemma 2 can be applied to yrw and zsw. To this end, we pick r and s to be the intersections of T_{uw} and the lines through w such that $\angle awr = \angle bws = 5\theta/4$, i.e. the line through w such that $w \in C_i^v$ with either $1 \leq i \leq k$, or $i = k$ and $|cw| \leq |dw|$. Next, it remains to handle Case (c), where v lies in raw.

Case (c): Vertex v lies in raw. Let c and d be the upper and lower right corners of T_{vw} and let a'' be the intersection of aw and the line through v, parallel to ua (see Figure 6c). Let p be the intersection of the left boundary of T_{uw} and the line through w such that $\angle awp = 3\theta/4$. Let β be $\angle a''wv$ and let γ be the angle between vw and the bisector of T_{vw}. We split this case into two subcases, depending on the location of v: (i) v lies below pw, (ii) v lies above pw. Since T_{vw} is smaller than T_{uw}, the induction hypothesis gives a bound on $\delta(v, w)$. Since $|uv| \leq |ua'| + |a'v|$ and v, a'', a, and a' form a parallelogram, we need to show that $\delta(v, w) \leq |va''| + |a''w| \cdot \boldsymbol{c}$ for both cases in order to complete the proof.

Case (i): When v lies below pw, w lies below the bisector of T_{vw} and the induction hypothesis for T_{vw} gives $\delta(v, w) \leq |vc| + |cw| \cdot \boldsymbol{c}$. We note that $\gamma = 5\theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(5\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(5\theta/4 - \beta))$. As this function is decreasing in β for $3\theta/4 \leq \beta \leq \theta$, it is maximized when β equals $3\theta/4$. Hence $\boldsymbol{c}$ needs to be at least $\boldsymbol{c} \geq (\cos(\theta/2) - \sin(3\theta/4))/(\cos(\theta/4) - \sin\theta)$, which is less than $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (ii): When v lies above pw, the induction hypothesis for T_{vw} gives $\delta(v, w) \leq \max\{|vc| + |cw| \cdot \boldsymbol{c}, |vd| + |dw| \cdot \boldsymbol{c}\}$. If $\delta(v, w) \leq |vc| + |cw| \cdot \boldsymbol{c}$, we note that $\gamma = \theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\theta/4 - \beta) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 - \beta))$. As this function is decreasing in β for $0 \leq \beta \leq \theta/4$, it is maximized when β equals 0. Hence $\boldsymbol{c}$ needs to be at least $\boldsymbol{c} \geq \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

If $\delta(v, w) \leq |vd| + |dw| \cdot \boldsymbol{c}$, we note that $\gamma = \theta/4 + \beta$. Hence Lemma 3 gives that the inequality holds when $\boldsymbol{c} \geq (\cos(\beta - \theta/4) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(\theta/4 + \beta))$, which is equal to $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$. $\square$