# On Plane Constrained Bounded-Degree Spanners*

Prosenjit Bose[1], Rolf Fagerberg[2], André van Renssen[1], Sander Verdonschot[1]

[1] School of Computer Science, Carleton University, Ottawa, Canada.
Email: jit@scs.carleton.ca, avrensse@connect.carleton.ca,
sander@cg.scs.carleton.ca.
[2] Dept. of Mathematics and Computer Science, University of Southern Denmark.
Email: rolf@imada.sdu.dk

**Abstract.** Let $P$ be a set of points in the plane and $S$ a set of non-crossing line segments with endpoints in $P$. The visibility graph of $P$ with respect to $S$, denoted $\mathit{Vis}(P,S)$, has vertex set $P$ and an edge for each pair of vertices $u, v$ in $P$ for which no line segment of $S$ properly intersects $uv$. We show that the constrained half-$\theta_6$-graph (which is identical to the constrained Delaunay graph whose empty visible region is an equilateral triangle) is a plane 2-spanner of $\mathit{Vis}(P,S)$. We then show how to construct a plane 6-spanner of $\mathit{Vis}(P,S)$ with maximum degree $6+c$, where $c$ is the maximum number of segments adjacent to a vertex.

## 1 Introduction

A Euclidean geometric graph $G$ is a graph whose vertices are points in the plane and whose edges are line segments between pairs of points. Edges are weighted by their Euclidean length. The distance between two vertices $u$ and $v$ in $G$, denoted by $d_G(u,v)$ or simply $d(u,v)$, is defined as the length of the shortest path between $u$ and $v$ in $G$. A subgraph $H$ of $G$ is a $t$-spanner of $G$ (for $t \geq 1$) if for each pair of vertices $u$ and $v$, $d_H(u,v) \leq t \cdot d_G(u,v)$. The value $t$ is the *spanning ratio* or *stretch factor*. The graph $G$ is referred to as the *underlying graph* of the $t$-spanner $H$. The spanning properties of various geometric graphs have been studied extensively in the literature (see [6] for a comprehensive overview of the topic). However, most of the research has focused on constructing spanners where the underlying graph is the complete Euclidean geometric graph. We study this problem in a more general setting with the introduction of line segment *constraints*.

Specifically, let $P$ be a set of points in the plane and let $S$ be a set of *constraints* such that each constraint is a line segment between two vertices in $P$. The set of constraints is planar, i.e. no two constraints intersect properly. Two vertices $u$ and $v$ can see each other if and only if either the line segment $uv$ does not properly intersect any constraint or $uv$ is itself a constraint. If two vertices

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$u$ and $v$ can see each other, the line segment $uv$ is a *visibility edge*. The *visibility graph* of $P$ with respect to a set of constraints $S$, denoted $Vis(P, S)$, has $P$ as vertex set and all visibility edges as edge set. In other words, it is the complete graph on $P$ minus all edges that properly intersect one or more constraints in $S$.

This setting has been studied extensively within the context of motion planning amid obstacles. Clarkson [4] was one of the first to study this setting in this context and showed how to construct a linear-sized $(1+\epsilon)$-spanner of $Vis(P, S)$. Subsequently, Das [5] showed how to construct a spanner of $Vis(P, S)$ with constant spanning ratio and constant degree. Bose and Keil [3] showed that the Constrained Delaunay Triangulation is a 2.42-spanner of $Vis(P, S)$. In this article, we show that the constrained half-$\theta_6$-graph (which is identical to the constrained Delaunay graph whose empty visible region is an equilateral triangle) is a plane 2-spanner of $Vis(P, S)$. A difficulty in proving the latter stems from the fact that the constrained Delaunay graph is **not** necessarily a triangulation. We then generalize the elegant construction of Bonichon *et al.* [2] to show how to construct a plane 6-spanner of $Vis(P, S)$ with maximum degree $6 + c$, where $c = \max\{c(v) | v \in P\}$ and $c(v)$ is the number of constraints incident to $v$.

## 2 Preliminaries

We define a *cone* $C$ to be the region in the plane between two rays originating from a vertex referred to as the apex of the cone. We let six rays originate from each vertex, with angles to the positive $x$-axis being multiples of $\pi/3$ (see Fig. 1). Each pair of consecutive rays defines a cone. For ease of exposition, we only consider point sets in general position: no two points define a line parallel to one of the rays that define the cones and no three points are collinear. These assumptions imply that we can consider the cones to be open.

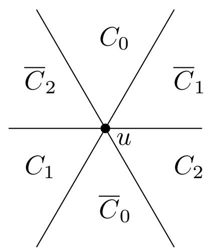


**Fig. 1.** The cones having apex $u$

Let $(\overline{C}_1, C_0, \overline{C}_2, C_1, \overline{C}_0, C_2)$ be the sequence of cones in counterclockwise order starting from the positive $x$-axis. The cones $C_0$, $C_1$, and $C_2$ are called *positive* cones and $\overline{C}_0$, $\overline{C}_1$, and $\overline{C}_2$ are called *negative* cones. By using addition and subtraction modulo 3 on the indices, positive cone $C_i$ has negative cone $\overline{C}_{i+1}$ as clockwise next cone and negative cone $\overline{C}_{i-1}$ as counterclockwise next cone. A similar statement holds for negative cones. We use $C_i^u$ and $\overline{C}_j^u$ to denote cones $C_i$ and $\overline{C}_j$ with apex $u$. Note that for any two vertices $u$ and $v$, $v \in C_i^u$ if and only if $u \in \overline{C}_i^v$.

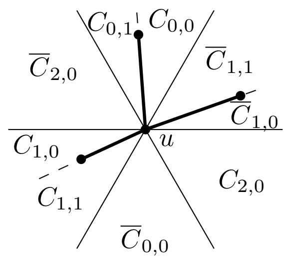


**Fig. 2.** The subcones having apex $u$. Constraints are shown as thick line segments.

Let vertex $u$ be an endpoint of a constraint $c$ and let the other endpoint $v$ lie in cone $C_i^u$. The lines through all such constraints $c$ split $C_i^u$ into several parts. We call these parts *subcones* and denote the $j$-th subcone of $C_i^u$ by $C_{i,j}^u$, numbered in counterclockwise order. When a constraint $c = (u, v)$ splits a cone of $u$ into two subcones, we define $v$ to lie in both of these subcones. We call a subcone of a positive cone a positive subcone and a subcone of a negative cone a negative subcone. We consider a cone that is not split as its own single subcone.

We now introduce the constrained half-$\theta_6$-graph, a generalized version of the half-$\theta_6$-graph as described by Bonichon *et al.* [1]: for each positive subcone of each vertex $u$, add an edge from $u$ to the closest vertex in that subcone that can see $u$, where distance is measured along the bisector of the original cone (not the subcone). More formally, we add an edge between two vertices $u$ and $v$ if $v$ can see $u$, $v \in C_{i,j}^u$, and for all points $w \in C_{i,j}^u$ that can see $u$ ($v \neq w$), $|uv'| \leq |uw'|$, where $v'$ and $w'$ denote the projection of $v$ and $w$ on the bisector of $C_i^u$, respectively, and $|xy|$ denotes the length of the line segment between two points $x$ and $y$. Note that our assumption of general position implies that each vertex adds at most one edge to the graph for each of its positive subcones.

Given a vertex $w$ in a positive cone $C_i^u$ of vertex $u$, we define the *canonical triangle* $T_{uw}$ to be the triangle defined by the borders of $C_i^u$ and the line through $w$ perpendicular to the bisector of $C_i^u$. Note that for each pair of vertices there exists a unique canonical triangle. We say that a region is *empty* if it does not contain any vertices of $P$.

## 3 Spanning Ratio of the Constrained Half-$\theta_6$-Graph

In this section we show that the constrained half-$\theta_6$-graph is a plane 2-spanner of the visibility graph. To do this, we first mention a property of visibility graphs.

**Lemma 1.** *Let $u$, $v$, and $w$ be three arbitrary points in the plane such that $uw$ and $vw$ are visibility edges and $w$ is not the endpoint of a constraint intersecting the interior of triangle $uvw$. Then there exists a convex chain of visibility edges from $u$ to $v$ in triangle $uvw$, such that the polygon defined by $uw$, $wv$ and the convex chain is empty.*

**Theorem 1.** *The constrained half-$\theta_6$-graph is a 2-spanner of the visibility graph.*

*Proof.* Given two vertices $u$ and $w$ such that $uw$ is a visibility edge, we assume w.l.o.g. that $w \in C_{0,j}^u$. We prove that $\delta(u, w) \leq 2 \cdot |uw|$, where $\delta(x, y)$ denotes the length of the shortest path from $x$ to $y$ inside $T_{xy}$ in the constrained half-$\theta_6$-graph. We prove this by induction on the area of $T_{uw}$ (formally, induction on the rank, when ordered by area, of the triangles $T_{xy}$ for all pairs of vertices $x$ and $y$ that can see each other). Let $a$ and $b$ be the upper left and right corner of $T_{uw}$, and let $A$ and $B$ be the triangles $uaw$ and $ubw$, respectively (see Fig. 3). Our inductive hypothesis is the following: If $A$ is empty, then $\delta(u, w) \leq |ub| + |bw|$. If $B$ is empty, then $\delta(u, w) \leq |ua| + |aw|$. If neither $A$ nor $B$ is empty, then $\delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\}$.

We first note that this induction hypothesis implies the theorem: using the side of $T_{uw}$ as the unit of length, we have that $\delta(u, w) \leq (\sqrt{3} \cdot \cos\alpha + \sin\alpha) \cdot |uw|$, where $\alpha$ is the unsigned angle between $uw$ and the bisector of $C_0^u$. This expression is increasing for $\alpha \in [0, \pi/6]$. Inserting the extreme value $\pi/6$ yields a spanning ratio of 2.

**Base case:** Triangle $T_{uw}$ has minimal area. Since the triangle is a smallest canonical triangle, $w$ is the closest vertex to $u$ in its positive subcone. Hence the edge $(u, w)$ must be in the constrained half-$\theta_6$-graph, and $\delta(u, w) = |uw|$. From the triangle inequality, we have that $|uw| \leq \min\{|ua| + |aw|, |ub| + |bw|\}$, so the induction hypothesis holds.

**Induction step:** We assume that the induction hypothesis holds for all pairs of vertices that can see each other and have a canonical triangle whose area is smaller than the area of $T_{uw}$. If $(u, w)$ is an edge in the constrained half-$\theta_6$-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between $u$ and $w$, let $v_0$ be the vertex closest to $u$ in the positive subcone containing $w$, and let $a_0$ and $b_0$ be the upper left and right corner of $T_{uv_0}$, respectively (see Fig. 3). By definition, $\delta(u, w) \leq |uv_0| + \delta(v_0, w)$, and by the triangle inequality, $|uv_0| \leq \min\{|ua_0| + |a_0v_0|, |ub_0| + |b_0v_0|\}$. We assume w.l.o.g. that $v_0$ lies to the left of $uw$, which means that $A$ is not empty.

Let $x$ be the intersection of $uw$ and $a_0b_0$. By definition $x$ can see $u$ and $w$. Since $v_0$ is the closest visible vertex to $u$, $v_0$ can see $x$ as well. Otherwise Lemma 1 would give us a convex chain of vertices connecting $v_0$ to $x$, all of which would be closer and able to see $u$. By applying Lemma 1 to triangle $v_0xw$, a convex chain $v_0, ..., v_k = w$ of visibility edges connecting $v_0$ and $w$ exists (see Fig. 3).

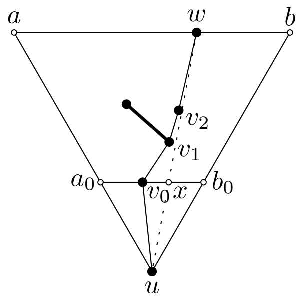


**Fig. 3.** A convex chain from $v_0$ to $w$

When looking at two consecutive vertices $v_{i-1}$ and $v_i$ along the convex chain, there are three types of configurations: (i) $v_{i-1} \in C_1^{v_i}$, (ii) $v_i \in C_0^{v_{i-1}}$ and $v_i$ lies to the right of $v_{i-1}$, (iii) $v_i \in C_0^{v_{i-1}}$ and $v_i$ lies to the left of $v_{i-1}$. Let $A_i = v_{i-1}a_iv_i$ and $B_i = v_{i-1}b_iv_i$, the vertices $a_i$ and $b_i$ will be defined for each case. By convexity, the direction of $\overrightarrow{v_iv_{i+1}}$ is rotating counterclockwise for increasing $i$. Thus, these configurations occur in the order Type (i), Type (ii), and Type (iii) along the convex chain from $v_0$ to $w$. We bound $\delta(v_{i-1}, v_i)$ as follows:

**Type (i):** If $v_{i-1} \in C_1^{v_i}$, let $a_i$ and $b_i$ be the upper left and lower corner of $T_{v_iv_{i-1}}$, respectively. Triangle $B_i$ lies between the convex chain and $uw$, so it must be empty. Since $v_i$ can see $v_{i-1}$ and $T_{v_iv_{i-1}}$ has smaller area than $T_{uw}$, the induction hypothesis gives that $\delta(v_{i-1}, v_i)$ is at most $|v_{i-1}a_i| + |a_iv_i|$.

**Type (ii):** If $v_i \in C_0^{v_{i-1}}$, let $a_i$ and $b_i$ be the left and right corner of $T_{v_{i-1}v_i}$, respectively. Since $v_i$ can see $v_{i-1}$ and $T_{v_{i-1}v_i}$ has smaller area than $T_{uw}$, the induction hypothesis applies. Whether $A_i$ and $B_i$ are empty or not, $\delta(v_{i-1}, v_i)$ is at most $\max\{|v_{i-1}a_i| + |a_iv_i|, |v_{i-1}b_i| + |b_iv_i|\}$. Since $v_i$ lies to the right of $v_{i-1}$, we know $|v_{i-1}a_i| + |a_iv_i| > |v_{i-1}b_i| + |b_iv_i|$, so $\delta(v_{i-1}, v_i)$ is at most $|v_{i-1}a_i| + |a_iv_i|$.

**Type (iii):** If $v_i \in C_0^{v_{i-1}}$ and $v_i$ lies to the left of $v_{i-1}$, let $a_i$ and $b_i$ be the left and right corner of $T_{v_{i-1}v_i}$, respectively. Since $v_i$ can see $v_{i-1}$ and $T_{v_{i-1}v_i}$

has smaller area than $T_{uw}$, we can apply the induction hypothesis. Thus, if $B_i$ is empty, $\delta(v_{i-1}, v_i)$ is at most $|v_{i-1}a_i| + |a_i v_i|$ and if $B_i$ is not empty, $\delta(v_{i-1}, v_i)$ is at most $|v_{i-1}b_i| + |b_i v_i|$.

To complete the proof, we consider three cases: (a) $\angle awu \leq \pi/2$, (b) $\angle awu > \pi/2$ and $B$ is empty, (c) $\angle awu > \pi/2$ and $B$ is not empty.

**Case (a):** If $\angle awu \leq \pi/2$, the convex chain cannot contain any Type (iii) configurations. We can now bound $\delta(u, w)$ by using these bounds (see Fig. 4): $\delta(u, w) \leq |uv_0| + \sum_{i=1}^{k} \delta(v_{i-1}, v_i) \leq |ua_0| + |a_0 v_0| + \sum_{i=1}^{k}(|v_{i-1}a_i| + |a_i v_i|)$. We see that the latter is equal to $|ua| + |aw|$ as required.

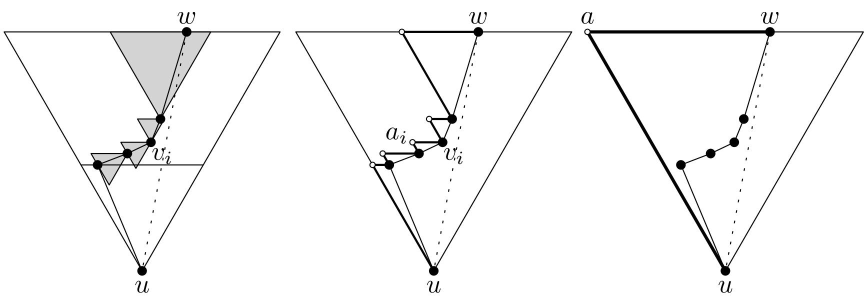


**Fig. 4.** Visualization of the paths (thick lines) in the inequalities of case (a)

**Case (b):** If $\angle awu > \pi/2$ and $B$ is empty, the convex chain can contain Type (iii) configurations. However, since $B$ is empty and the area between the convex chain and $uw$ is empty (by Lemma 1), all $B_i$ are also empty. Using the computed bounds on the lengths of the paths between the points along the convex chain, we can bound $\delta(u, w)$ as in the previous case.

**Case (c):** If $\angle awu > \pi/2$ and $B$ is not empty, the convex chain can contain Type (iii) configurations and since $B$ is not empty, the triangles $B_i$ need not be empty. Recall that $v_0$ lies in $A$, hence neither $A$ nor $B$ are empty. Therefore, it suffices to prove that $\delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\} = |ub| + |bw|$. Let $T_{v_j v_{j+1}}$ be the first Type (iii) configuration along the convex chain (if it has any), let $a'$ and $b'$ be the upper left and right corner of $T_{uv_j}$, and let $b''$ be the upper right corner of $T_{v_j w}$ (see Fig. 5).

$$
\begin{aligned}
\delta(u, w) &\leq |uv_0| + \sum_{i=1}^{k} \delta(v_{i-1}, v_i) & (1)\\
&\leq |ua_0| + |a_0 v_0| + \sum_{i=1}^{j}(|v_{i-1}a_i| + |a_i v_i|) + \sum_{i=j+1}^{k}(|v_{i-1}b_i| + |b_i v_i|) & (2)\\
&= |ua'| + |a'v_j| + |v_j b''| + |b''w| & (3)\\
&\leq |ub'| + |b'v_j| + |v_j b''| + |b''w| & (4)\\
&= |ub| + |bw| & (5)
\end{aligned}
$$

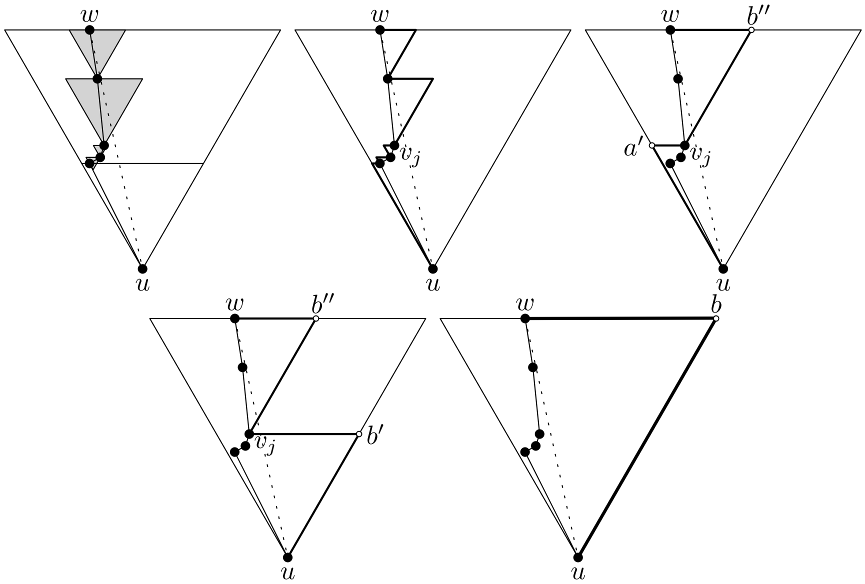


**Fig. 5.** Visualization of the paths (thick lines) in the inequalities of case (c)

□

Next, we prove that the constrained half-$\theta_6$-graph is plane.

**Lemma 2.** *Let $u$, $v$, $x$, and $y$ be four distinct vertices such that the two canonical triangles $T_{uv}$ and $T_{xy}$ intersect. Then at least one of the corners of one triangle is contained in the other triangle.*

**Lemma 3.** *The constrained half-$\theta_6$-graph is plane.*

*Proof.* Assume that two edges $uv$ and $xy$ cross at a point $p$. Since the two edges are contained in their canonical triangles, these must intersect. By Lemma 2 we know that at least one of the corners of one triangle lies inside the other. Assume w.l.o.g. that the upper right corner of $T_{xy}$ lies inside $T_{uv}$. Since $uv$ and $xy$ cross, this also means that either $x$ or $y$ must lie in $T_{uv}$.

Assume w.l.o.g. that $v \in C^u_{0,j}$ and $y \in T_{uv}$. If $y \in C^u_{0,j}$, we look at triangle $upy$. Using that both $u$ and $y$ can see $p$, we get by Lemma 1 that either $u$ can see $y$ or $upy$ contains a vertex. In both cases, $u$ can see a vertex in this subcone that is closer than $v$, contradicting the existence of the edge $uv$.

If $y \notin C^u_{0,j}$, there exists a constraint $uz$ such that $v$ lies to one side of the line through $uz$ and $y$ lies on the other side. Since this constraint cannot cross $yp$, $z$ lies inside $upy$ and is therefore closer to $u$ than $v$. Since by definition $z$ can see $u$, this also contradicts the existence of $uv$. □

## 4 Bounding the Maximum Degree

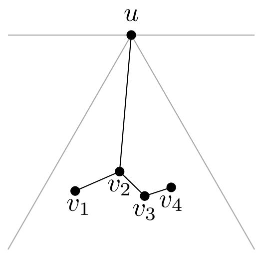


**Fig. 6.** The edges that are added to $G_9(P)$ for a negative subcone of a vertex $u$ with canonical sequence $v_1, v_2, v_3$ and $v_4$

In this section, we show how to construct a bounded degree subgraph $G_9(P)$ of the constrained half-$\theta_6$-graph that is a 6-spanner of the visibility graph. Given a vertex $u$ and one of its negative subcones, we define the *canonical sequence* of this subcone as the counterclockwise order of the vertices in this subcone that are neighbors of $u$ in the constrained half-$\theta_6$-graph (see Fig. 6). The *canonical path* is defined by connecting consecutive vertices in the canonical sequence. This definition differs slightly from the one used by Bonichon *et al.* [2].

To construct $G_9(P)$, we start with a graph with vertex set $P$ and no edges. Then for each negative subcone of each vertex $u \in P$, we add the canonical path and an edge between $u$ and the closest vertex along this path, where distance is measured using the projections of the vertices along the bisector of the cone containing the subcone. This construction is similar to the construction of the unconstrained degree-9 half-$\theta_6$-graph described by Bonichon *et al.* [2]. Note that since every edge of the canonical path is part of the constrained half-$\theta_6$-graph, $G_9(P)$ is a subgraph of the constrained half-$\theta_6$-graph. We proceed to prove that $G_9(P)$ is a spanning subgraph of the constrained half-$\theta_6$-graph with spanning ratio 3.

**Theorem 2.** *$G_9(P)$ is a 3-spanner of the constrained half-$\theta_6$-graph.*

*Proof.* We prove the lemma by showing that for each edge $uw$ of the constrained half-$\theta_6$-graph $H$ that is not part of $G_9(P)$, $d_{G_9(P)}(u, w) \leq 3 \cdot d_H(u, w)$.

We assume w.l.o.g. that $w \in \overline{C}_0^u$. Let $v_0$ be the vertex closest to $u$ on the canonical path and let $v_0, v_1, ..., v_k = w$ be the vertices along the canonical path from $v_0$ to $w$ (see Fig. 7). Let $l_j$ and $r_j$ denote the rays defining the counterclockwise and clockwise boundaries of $C_0^{v_j}$ for $0 \leq j \leq k$ and let $r$ denote the ray defining the clockwise boundary of $\overline{C}_0^u$. Let $m_j$ be the intersection of $l_j$ and $r_{j-1}$, for $1 \leq j \leq k$, and let $m_0$ be the intersection of $l_0$ and $r$. Let $w'$ be the intersection of $r$ and the horizontal line through $w$ and let $w''$ be the intersection of $l_k$ and $r$. The length of the path between $u$ and $w$ in $G_9(P)$ can now be bounded as follows:

$$
\begin{aligned}
d_{G_9(P)}(u, w) &\leq |uv_0| + \sum_{j=1}^{k} |v_{j-1}v_j| && (6)\\
&\leq |um_0| + \sum_{j=0}^{k} |m_j v_j| + \sum_{j=0}^{k-1} |v_j m_{j+1}| && (7)\\
&= |um_0| + |ww''| + |m_0 w''| && (8)\\
&\leq |uw'| + 2 \cdot |ww'| && (9)
\end{aligned}
$$

Let $\alpha$ be $\angle w'uw$ and let $x$ be the intersection of $uw'$ and the line through $w$ perpendicular to $uw'$. Using some basic trigonometry, we get $|uw'| = |uw| \cdot \cos\alpha + |uw| \cdot \sin\alpha/\sqrt{3}$ and $|ww'| = 2 \cdot |uw| \cdot \sin\alpha/\sqrt{3}$. Thus the spanning ratio can be expressed as:

$$\frac{d_{G_9(P)}(u,w)}{|uw|} \leq \cos\alpha + 5 \cdot \frac{\sin\alpha}{\sqrt{3}} \tag{10}$$

Since this is a non-decreasing function on $0 < \alpha \leq \pi/3$, its maximum value is obtained when $\alpha = \pi/3$, where the spanning ratio is 3. □

It follows from Theorems 1 and 2 that $G_9(P)$ is a 6-spanner of the visibility graph.

**Corollary 1.** *$G_9(P)$ is a 6-spanner of the visibility graph.*

Now, we bound the maximum degree.

**Lemma 4.** *When a vertex $v$ has at least two constraints in the same positive cone $C_i^v$, the closest vertex $u$ between two consecutive constraints has $v$ as the closest vertex in the subcone of $\overline{C}_i^u$ that contains $v$ and $v$ is the only vertex on the canonical path of this subcone.*

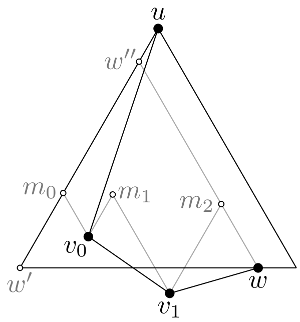

**Fig. 7.** Bounding the length of the canonical path

*Proof.* Since $u$ is the closest vertex in this positive subcone $C_{i,j}^v$, we know $\overline{C}_i^u \cap C_{i,j}^v$ contains only vertices $u$ and $v$. Hence $v$ is the only visible vertex in $\overline{C}_i^u \cap C_{i,j}^v$ and $v$ is the only vertex along the canonical path of $\overline{C}_i^u \cap C_{i,j}^v$. □

**Lemma 5.** *Every vertex $v$ in $G_9(P)$ has degree at most $c(v) + 9$.*

*Proof.* To bound the degree of a vertex, we use a charging scheme that charges the edges added during the construction to the (sub)cones of that vertex. We prove that each positive cone has charge at most $c_i(v) + 2$ and each negative cone has charge at most $c_{\bar{i}}(v) + 1$, where $c_i(v)$ and $c_{\bar{i}}(v)$ are the number of constraints in the $i$-th positive and negative cone, respectively. Since a vertex has three positive and three negative cones and the $c_i(v)$ and $c_{\bar{i}}(v)$ sum up to $c(v)$, this implies that the total degree of a vertex is at most $c(v) + 9$. In fact, we will show that a positive cone is charged at most $\max\{2, c_i(v) + 1\}$.

We look at the canonical path in $\overline{C}_{i,j}^u$, created by a vertex $u$. We use $v$ to indicate an arbitrary vertex along the canonical path. Let $v'$ be the closest vertex along the canonical path and let $C_{i,k}^{v'}$ be the cone of $v$ that contains $u$. The edges of $G_9(P)$ are charged as follows (see Fig. 8):

- The edge $uv'$ is charged to $\overline{C}^u_{i,j}$ and to $C^{v'}_{i,k}$
- An edge of the canonical path that lies in $C^v_{i\pm1}$ is charged to $\overline{C}^v_{i\mp1}$
- An edge of the canonical path that lies in $\overline{C}^v_{i\pm1}$ is charged to $C^v_{i,k}$

Note that each edge is charged once to each of its endpoints.

We first prove that each positive cone has charge at most $\max\{2, c_i(v) + 1\}$. If the positive cone does not contain any constraints, a positive cone of a vertex $v$ containing $u$ is charged by the edge in that cone if $v$ is the closest visible vertex to $u$ and it is charged by the two adjacent negative cones if the edges of the canonical path lie in those cones. Note that since all charges are shifted one cone towards the positive cone containing $u$, other canonical paths cannot charge this positive cone of $v$. Also note that the positive cone is charged at most 2 if $v$ is not the closest vertex to $u$.

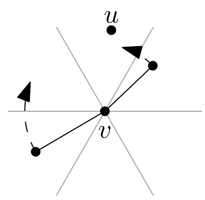


**Fig. 8.** Two edges of a canonical path and the associated charges

If $v$ is the closest vertex to $u$, the negative cones adjacent to this positive cone cannot contain any vertices of the canonical path, since these vertices would be closer to $u$ than $v$ is. Hence, if $v$ is the closest vertex to $u$, the positive cone containing $u$ is charged 1.

If the cone contains constraints, we use Lemma 4 to get a charge of at most $c_i(v) - 1$ in total for all subcones except the first and last one. We prove that these subcones can be charged at most 1 each.

We look at the first one. The only way to charge this subcone 2 is if $v$ is the closest vertex to $u$ in this subcone and the adjacent negative cone contains an edge to a vertex that is part of the same canonical path. But if $v$ is the closest vertex to $u$, the negative cones adjacent to this positive cone cannot contain any vertices of the canonical path, since these would be closer to $u$ than $v$ is. Hence, if $v$ is the closest vertex to $u$, the positive cone containing $u$ is charged 1. Therefore each positive cone has charge at most $\max\{2, c_i(v) + 1\}$.

Next, we prove that each negative cone has charge at most $c_{\bar{i}}(v) + 1$. A negative cone of a vertex $v$ is charged by the edge to the closest vertex in each of its subcones and it is charged by the two adjacent positive cones if the edges of the canonical paths lie in those cones (see Fig. 9). Suppose that $w$ lies in a positive cone of $v$ and $vw$ is part of the canonical path of $u$. Then $w$ lies in a negative cone of $u$, which means that $u$ lies in a positive cone of $w$ and cannot be part of a canonical path for $w$. Thus every negative cone can be charged by only one edge in an adjacent positive cone.

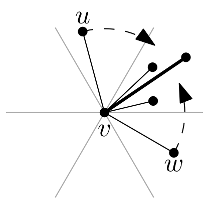


**Fig. 9.** If $vw$ is present, the negative cone does not contain edges having $v$ as endpoint.

If this negative cone does not contain any constraints, it remains to show that if one of $uv$ and $vw$ is present, the negative cone does not have an edge to the closest vertex in that cone. We assume w.l.o.g. that $vw$ is present, $u \in C^v_i \cap C^w_i$, and $w \in C^v_{i-1}$. Since $v$ and $w$ are neighbors on the canonical path, we know that the triangle $uvw$ is part of the constrained half-$\theta_6$-graph and it is empty. Furthermore, since the constrained half-$\theta_6$-graph is plane and $uw$ is an edge of the constrained half-$\theta_6$-graph, $v$ cannot have an edge to the closest vertex beyond $uw$. Hence the negative cone does not have an edge to the closest vertex in that cone.

Using a similar argument it can be shown that if one of $uv$ and $vw$ is present, the negative cone does not contain any constraints. Thus the charge of a negative cone is at most $c_{\bar{i}}(v) + 1$. □

**Corollary 2.** *If a positive cone has charge $c_i(v) + 2$, it is charged for two edges in the adjacent negative cones and does not contain any constraints having $v$ as an endpoint.*

### 4.1 Bounding the Maximum Degree Further

Using Corollary 2, we know that the only situation we need to modify to get the degree bound down to $c(v) + 6$ is the case where a positive cone is charged for two edges in the adjacent negative cones and does not contain any constraints (see Fig. 10).

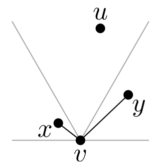


**Fig. 10.** A positive cone having charge 2

If neither $x$ nor $y$ is the vertex closest to $v$ in their respective cone, we do the following transformation on $G_9(P)$. First, we add an edge between $x$ and $y$. Next, we look at the sequence of vertices between $v$ and the closest vertex along the canonical path. If this sequence includes $x$, we remove $vy$. Otherwise we remove $vx$.

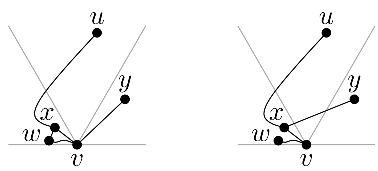


**Fig. 11.** Constructing $G_6(P)$ (right) from $G_9(P)$ (left)

We assume w.l.o.g. that $vy$ is removed. We look at vertex $w$, the neighbor of vertex $x$ on the canonical path of vertex $v$ containing $x$. Since $x$ is not the closest vertex to $v$, this vertex $w$ must exist. The edge $xw$ is removed if $w$ lies in a negative cone of $x$ and $w$ is not the closest vertex in this cone. The resulting graph is $G_6(P)$ (see Fig. 11). It can be shown that the newly added edges do not intersect each other, the constraints and the remaining edges of $G_9(P)$, which implies that $G_6(P)$ is plane. Before we prove that this construction yields a graph of maximum degree $6 + c$, we first show that the resulting graph is still a 3-spanner of the constrained half-$\theta_6$-graph.

**Lemma 6.** *Let $vx$ be an edge of $G_9(P)$ and let $x$ lie in a negative cone $\overline{C}_i^v$ of $v$. If $x$ is not the vertex closest to $v$ in $\overline{C}_i^v$, then the edge $vx$ is used by at most one canonical path.*

*Proof.* We prove the lemma by contradiction. Given that $x$ is not the vertex closest to $v$ in $\overline{C}_i^v$, assume that edge $vx$ is part of two canonical paths. This means that there exist vertices $u \in C_{i+1}^v \cap C_{i+1}^x$ and $w \in C_{i-1}^v \cap C_{i-1}^x$ such that $v$ and $x$ are neighbors on a canonical path of $u$ and $w$. Thus vertices $uvx$ and $wvx$ form two triangles in the constrained half-$\theta_6$-graph. By planarity, this implies that $vx$ is the only edge of $v$ in $\overline{C}_i^v$ (see Fig. 12). This implies that $x$ is the vertex closest to $v$ in $\overline{C}_i^v$. □

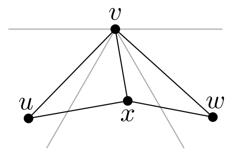


**Fig. 12.** Edge $vx$ is part of two canonical paths.

**Lemma 7.** *$G_6(P)$ is a 3-spanner of the half-$\theta_6$-graph.*

*Proof.* Since $G_9(P)$ is a 3-spanner of the constrained half-$\theta_6$-graph, we need to look only at the edges that were removed from this graph. Let $v$ be a vertex such that positive cone $C_i^v$ has charge 2, let $u$ be the vertex whose canonical path charged 2 to $C_i^v$, and let $x \in \overline{C}_{i-1}^v$ and $y \in \overline{C}_{i+1}^v$ be the neighbors of $v$ on this canonical path. We assume w.l.o.g. that $vy$ is removed. Since this removal potentially affects the spanning ratio of any path using $vy$, we need to look at the spanning path between $v$ and $y$ and the spanning path between $u$ and any vertex on the canonical path that uses $vy$. Since $y$ is not the closest vertex to $v$, Lemma 6 tells us that no other canonical path is affected.

Since $y$ is not the closest vertex to $v$, there exists a spanning path between $v$ and $y$ that does not use $vy$. Since $|xy| \leq |xv| + |vy|$, the length of the spanning path between $u$ and any vertex on the canonical path that uses $vy$ is not increased. Thus removing $vy$ does not affect the spanning ratio.

Next, we look at the other type of edge that is removed. Let $w$ be the neighbor of vertex $x$ on the canonical path of vertex $v$ containing $x$. Edge $wx$ is removed if $w$ lies in $\overline{C}_i^x$ and $w$ is not the closest vertex in $\overline{C}_i^x$. Since $x$ is the last vertex on the canonical path of $v$, we need to look only at the spanning path between $x$ and $w$ and the spanning path between $v$ and $x$. Since $y$ is not the closest vertex to $v$, Lemma 6 tells us that no other canonical path is affected.

Since $w$ is not the closest vertex to $x$, there exists a spanning path between $x$ and $w$ that does not use $xw$. By Lemma 6, $vx$ is part of only one canonical path and hence it is present in $G_6(P)$. Thus there exists a spanning path between $x$ and $v$ and removing $xw$ does not affect the spanning ratio. □

**Lemma 8.** *Every vertex $v$ in $G_6(P)$ has degree at most $c(v) + 6$.*

*Proof.* To bound the degree of a vertex, we look at the charges of the vertices. We prove that after the transformation each positive cone has charge at most

$c_i(v) + 1$ and each negative cone has charge at most $c_{\bar{i}}(v) + 1$. This implies that the total degree of a vertex is at most $c(v) + 6$. Since the charge of the negative cones is already at most $c_{\bar{i}}(v) + 1$, we focus on positive cones having charge 2.

Let $v$ be a vertex such that one of its positive cones $C_i^v$ has charge 2, let $u$ be the vertex whose canonical path charged 2 to $C_i^v$, and let $x \in \overline{C}_{i-1}^v$ and $y \in \overline{C}_{i+1}^v$ be the neighbors of $v$ on this canonical path (see Fig. 10). If $x$ or $y$ is the vertex closest to $v$ in $\overline{C}_{i-1}^v$ or $\overline{C}_{i+1}^v$, this edge has been charged to both that negative cone and $C_i^v$. Hence we can remove the charge to $C_i^v$ while maintaining that the charge is an upper bound on the degree of $v$.

If neither $x$ nor $y$ is the closest vertex in $\overline{C}_{i-1}^v$ or $\overline{C}_{i+1}^v$, edge $xy$ is added. We assume w.l.o.g. that edge $vy$ is removed. Thus $vy$ need not be charged, decreasing the charge of $C_i^v$ to 1. Since $vy$ was charged to $\overline{C}_{i-1}^y$ and this charge is removed, we charge edge $xy$ to $\overline{C}_{i-1}^y$. Thus the charge of $y$ does not change.

It remains to show that we can charge $xy$ to $x$. We look at vertex $w$, the neighbor of $x$ on the canonical path of $v$ in $\overline{C}_{i-1}^v$. Since $x$ is not the closest vertex to $v$ in $\overline{C}_{i-1}^v$, the canonical path and vertex $w$ exist. Since vertices $uvx$ form a triangle in the constrained half-$\theta_6$-graph, $C_{i-1}^x$ has charge at most 1. Vertex $w$ can be in one of two cones with respect to $x$: $C_{i+1}^x$ and $\overline{C}_i^x$. If $w \in C_{i+1}^x$, $xw$ is charged to $\overline{C}_i^x$. Thus the charge of $C_{i-1}^x$ is 0 and we can charge $xy$ to it.

If $w \in \overline{C}_i^x$ and $w$ is the closest vertex to $x$ in $\overline{C}_i^x$, $xw$ has been charged to both $C_{i-1}^x$ and $\overline{C}_i^x$. We replace the charge of $C_{i-1}^x$ by $xy$ and the charge of $C_{i-1}^x$ remains 1. If $w \in \overline{C}_i^x$ and $w$ is not the closest vertex to $x$ in $\overline{C}_i^x$, $xw$ is removed. Since this edge was charged to $C_{i-1}^x$, we can charge $xy$ to $C_{i-1}^x$ and the charge of $C_{i-1}^x$ remains 1. □

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