Making triangulations 4-connected using flips

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Abstract

We show that any triangulation on n vertices can be transformed into a 4-connected one using at most $\lfloor (3n-6)/5 \rfloor$ edge flips. We also give an example of a triangulation that requires $\lceil (3n-10)/5 \rceil$ flips to be made 4-connected, showing that our bound is tight. Our result implies a new upper bound on the diameter of the flip graph of $5.2n - 24.4$, improving on the bound of $6n - 30$ by Mori *et al.* [4].

1 Introduction

Given a triangulation (a maximal planar simple graph) on a set of n vertices, we define an *edge flip* as removing an edge (a, b) from the graph and replacing it with the edge (c, d), where c and d are the other vertices of the triangles that had (a, b) as an edge. Figure 1 shows an example of an edge flip.

Flips have been studied mostly in two different settings: the *geometric* setting, where we are given a fixed set of points in the plane and edges are straight line segments, and the *combinatorial* setting, where we are only given the clockwise order of edges around each vertex (a combinatorial embedding). In this paper, we concern ourselves with the number of flips required to transform one triangulation into another in the combinatorial setting. We give a brief overview of previous work on this problem. A more detailed overview, including applications and related work, can be found in a survey by Bose and Hurtado [2].

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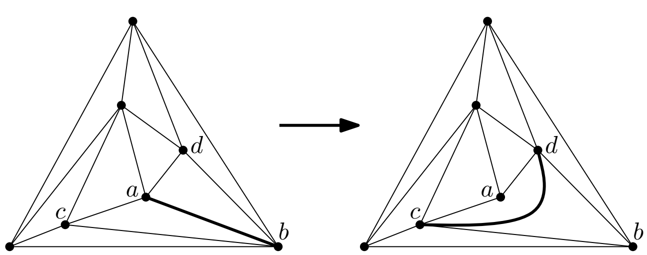

Figure 1: An example triangulation before and after flipping edge (a, b).

Given a set of n vertices, we can define its *flip graph* as the graph with a vertex for each distinct triangulation and an edge between two vertices if their corresponding triangulations differ by a single flip. Two triangulations are considered distinct if they are not isomorphic. In his seminal paper, Wagner [7] showed that there always exists a sequence of $O(n^2)$ flips that transforms a given triangulation into any other triangulation on the same set of vertices. In terms of the flip graph, Wagner showed that it is connected and has diameter $O(n^2)$. Komuro [3] was the first to show that the diameter is linear and Mori *et al.* [4] currently have the strongest upper bound of $6n - 30$.

The above results all show how to transform any triangulation into a fixed canonical triangulation. Transformation of one triangulation into another is then straightforward by transforming the first into the canonical triangulation and transforming the canonical triangulation into the second by reversing the sequence of flips for the second. Mori *et al.*'s algorithm to transform a triangulation into the canonical one consists of two steps. They first make the given triangulation 4-connected using at most $n-4$ flips. Since a 4-connected triangulation is always Hamiltonian [6], they then show how to transform this into the canonical triangulation by at most $2n-11$ flips, using a decomposition into two outerplanar graphs that share a Hamiltonian cycle as their outer faces.

The problem of making triangulations 4-connected has also been studied in the setting where many edges may be flipped simultaneously [1]. Bose *et al.* showed that any triangulation can be made 4-connected by one simultaneous flip and that $O(\log n)$ simultaneous flips are sufficient and sometimes necessary to transform between two given triangulations.

In Section 2, we show that any triangulation can be made 4-connected using at most $(3n-6)/5$ flips. This improves the first step of the construction by Mori *et al.* and results in a new upper bound on the diameter of the flip graph of $5.2n - 24.4$. Then we show in Section 3 that there are triangulations that require $(3n-10)/5$ flips to be made 4-connected. Since the difference with the upper bound is less than one flip, this bound is tight. Section 4 contains proofs for various technical lemmas that are necessary for the main result. Section 5 contains conclusions and future work.

2 Upper Bound

In this section we prove an upper bound on the number of flips required to make any given triangulation 4-connected. Specifically, we show that $(3n-6)/5$ flips always suffice. The proof references several technical lemmas whose proofs can be found in Section 4.

We are given a triangulation T, along with a combinatorial embedding specifying the clockwise order of edges around each vertex of T. In addition, one of the faces of T is marked as the *outer face*. A *separating triangle* D is a cycle in T of length three whose removal splits T into two non-empty connected components. We call the component that contains vertices of the outer face the *exterior* of D, and the other component the *interior* of D. A vertex in the interior of D is said to be *inside* D and likewise, a vertex in the exterior of D is *outside* D. An edge is inside a separating triangle if one of its endpoints is inside. A separating triangle A *contains* another separating triangle B if and only if the interior of B is a subgraph of the interior of A with a strictly smaller vertex set. If A contains B, A is called the *containing* triangle. A separating triangle that is contained by the largest number of separating triangles in T is called *deepest*. Since containment is transitive, a deepest separating triangle cannot contain any separating triangles, as these would have a higher number of containing triangles. Finally, we call an edge that does not belong to any separating triangle a *free edge*.

We will remove all separating triangles from T by repeatedly flipping an edge of a deepest separating triangle. This makes T 4-connected, as a triangulation is 4-connected if and only if it has no separating triangles. This technique was also used by Mori *et al.* [4], who proved the following lemma.

Lemma 1 *In a triangulation with $n \geq 6$ vertices, flipping any edge of a separating triangle D will remove that separating triangle. This never introduces a new separating triangle, provided that the selected edge belongs to multiple separating triangles or none of the edges of D belong to multiple separating triangles.*

Before we can prove our new upper bound, we need to prove another property of separating triangles.

Lemma 2 *In a triangulation T, every vertex v of a separating triangle D is incident to at least one free edge inside D.*

Proof. Consider one of the edges of D incident to v. Since D is separating, its interior cannot be empty and since D is part of T, there is a triangular face inside D that uses this edge. Now consider the other edge e of this face that is incident to v.

The remainder of the proof is by induction on the number of separating triangles contained in D. For the base case, assume that D does not contain any other separating triangles. Then e must be a free edge and we are done.

For the induction step, there are two further cases. If e does not belong to a separating triangle, we are again done, so assume that e belongs to a separating triangle D'. Since D' is itself a separating triangle contained in D and containment is transitive, the number of separating triangles contained by D' must be strictly smaller than that of D. Since v is also a vertex of D', our induction hypothesis tells us that there is a free edge incident to v inside D'. Since D' is contained in D, this edge is also inside D. $\square$

Theorem 3 *A triangulation on $n \geq 6$ vertices can be made 4-connected using at most $\lfloor (3n-6)/5 \rfloor$ flips.*

Proof. We prove this using a charging scheme. We begin by placing a coin on every edge of the triangulation. Then we flip an edge of a deepest separating triangle (preferring edges that belong to multiple separating triangles) until no separating triangles are left. We pay 5 coins for every flip. During this process, we maintain two invariants:

- Every edge of a separating triangle has a coin.
- Every vertex of a separating triangle has an incident free edge that is inside the triangle and has a coin.

These invariants have several nice properties. First, an edge can either be a free edge or belong to a separating triangle, but not both. So at any given time, only one invariant applies to an edge. Second, an edge only needs one coin to satisfy the invariants, even if it is on multiple separating triangles or is a free edge for multiple separating triangles. These two properties imply that the invariants hold initially, since by Lemma 2, every vertex of a separating triangle has an incident free edge. Third, flipping an edge that satisfies the criteria of Lemma 1 cannot upset the invariants, since its separating triangle is removed and no new separating triangles are introduced. Finally, since we pay 5 coins per flip and there are $3n-6$ edges, by placing a coin on each edge, we can flip at most $\lfloor (3n-6)/5 \rfloor$ edges.

Now let us take a closer look at the kind of edges we can use to pay for flipping an edge of a deepest separating triangle D. We identify four types of edges here:

Type 1 (■). The flipped edge e. By Lemma 1, e cannot belong to any separating triangle after the flip, so the first invariant still holds if we remove e's coin. Before the flip, e was not a free edge, so the second invariant was satisfied even without e's coin. Since the flip did not introduce any new separating triangles, this is still the case.

Type 2 (□). A non-flipped edge e of D that is not shared with any other separating triangle. By Lemma 1, the flip removed D and did not introduce any new separating triangles. Therefore e cannot belong to any separating triangle, so the first invariant still holds if we remove e's coin. By the same argument as for the previous type, e is also not required to have a coin to satisfy the second invariant.

Type 3 (○). A free edge e of a vertex of D that is not shared with any containing separating triangle. Since e did not belong to any separating triangle and the flip did not introduce any new ones, e is not required to have a coin to satisfy the first invariant. Further, since the flip removed D and e is not incident to a vertex of another separating triangle that contains it, it is no longer required to have a coin to satisfy the second invariant. Therefore we can remove its coin without violating either invariant.

Type 4 (●). A free edge e incident to a vertex v of D, where v is an endpoint of an edge e' of D that is shared with a non-containing separating triangle B, provided that we flip e'. Any separating triangle that contains D but not B must share e' (Lemma 10) and is therefore removed by the flip. So every separating triangle after the flip that contains D also contains B. In particular, this also holds for containing triangles that share v. Since the second invariant requires only one free edge with a coin for each vertex, we can safely charge the one inside D, as long as we do not charge the free edge in B.

To decide which edges we flip and how we pay for each flip, we distinguish five cases, based on the number of edges shared with other separating triangles and whether any of these triangles contain D. These cases are illustrated in Figures 2, 3, and 4.

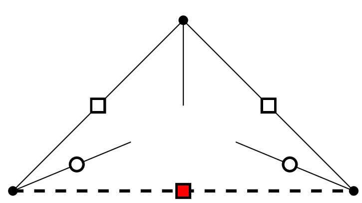

Figure 2: The edges that are charged if the deepest separating triangle does not share any edges with other separating triangles. The flipped edge is dashed and the charged edges are marked with red boxes (Type 1), white boxes (Type 2), white disks (Type 3) or red disks (Type 4).

Case 1. D does not share any edges with other separating triangles (Figure 2). In this case, we flip any of D's edges. By the first invariant, each edge of D has a coin. These edges all fall into Types 1 and 2 above, so we use their coins to pay for the flip. Further, D can share at most one vertex with a containing triangle (Lemma 8), so we charge two free edges, each incident to one of the other two vertices (Type 3).

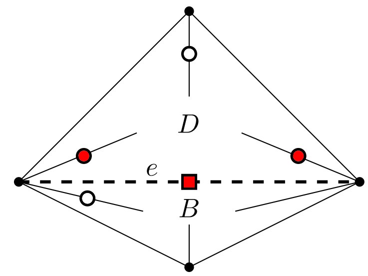

Figure 3: The edges that are charged if the deepest separating triangle only shares edges with non-containing separating triangles.

Case 2. D does not share any edge with a containing triangle, but shares one or more edges with non-containing separating triangles (Figure 3). In this case, we flip one of the shared edges e. We charge e (Type 1) and two free edges inside D that are incident to the vertices of e (Type 4). This leaves us with two more coins that we need to charge.

Let B be the non-containing separating triangle that shares e with D. We first show that B must be deepest. There can be no separating triangles that contain D but not B, as any such triangle would have to share e (Lemma 10) and D does not share any edge with a containing triangle. Therefore any triangle that contains D must contain B as well. Since D is contained in the maximal number of separating triangles, this holds for B as well. This means that B cannot contain any separating triangles and to satisfy the second invariant we only need to concern ourselves with triangles that contain both B and D.

Now consider the number of vertices of the quadrilateral formed by B and D that can be shared with containing triangles. Since D does not share an edge with a containing triangle, it can share at most one vertex with a containing triangle (Lemma 8). Now suppose that B shares an edge with a containing triangle. Then one of the vertices of this edge is part of D as well. Since the other two vertices are both part of D, they cannot be shared with containing triangles. If B does not share an edge with a containing triangle, it too can share at most one vertex with containing triangles. Thus, in both cases, at most two vertices of the quadrilateral can be shared with containing triangles and we charge two free edges, each incident to one of the other two vertices, for the last two coins (Type 3).

Case 3. D shares an edge with a containing triangle A and does not share the other edges with any separating triangle (Figure 4a). In this case, we flip the shared edge and charge all of D's edges, since one is the flipped edge (Type 1) and the others are not shared (Type 2).

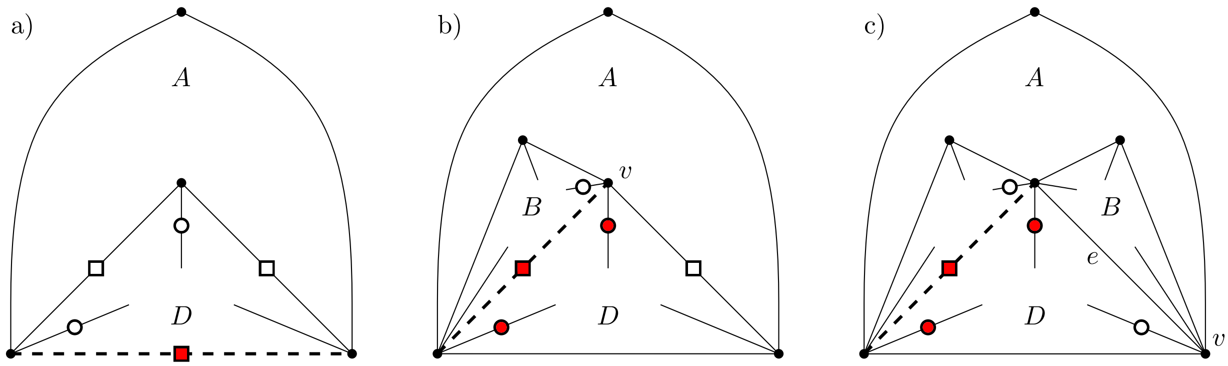

Figure 4: The edges that are charged if the deepest separating triangle shares an edge with a containing triangle.

The vertex of D that is not shared with A cannot be shared with any containing triangle (Lemma 9), so we charge a free edge incident to this vertex (Type 3).

Further, if A shares an edge with a containing triangle, it either shares the flipped edge, which means that the containing triangle is removed by the flip, or it shares another edge, in which case the vertex that is not an endpoint of this edge cannot be shared with any containing triangle. If A does not share an edge with a containing triangle, it can share at most one vertex with a containing triangle (Lemma 8). In both cases, one of the vertices of the flipped edge is not shared with any containing triangle (Type 3), so we charge a free edge incident to it.

Case 4. D shares an edge with a containing triangle A and one other edge with a non-containing separating triangle B (Figure 4b). In this case, we flip the edge that is shared with B. Let v be the vertex of D that is not shared with A. We charge the flipped edge (Type 1), the unshared edge of D (Type 2) and two free edges inside D that are incident to the vertices of the flipped edge (Type 4). We charge the last coin from a free edge in B that is incident to v. We can charge it, since v cannot be shared with a triangle that contains D (Lemma 9) and every separating triangle that contains B but not D must share the flipped edge as well (Lemma 10) and is therefore removed by the flip.

All that is left is to argue that there can be no separating triangle contained in B that requires the charge to satisfy the second invariant. Every separating triangle that contains D but not B must share the flipped edge (Lemma 10). Since D already shares another edge with a containing triangle and it cannot share two edges with containing triangles (Lemma 7), all separating triangles that contain D must also contain B. Since D is deepest, B must be deepest as well and therefore cannot contain any separating triangles.

Case 5. D shares one edge with a containing triangle A and the other two with non-containing separating triangles (Figure 4c). In this case we also flip the edge shared with one of the non-containing triangles. The charged edges are identical to the previous case, except that there is no unshared edge any more. Instead, we charge the last free edge in D.

Before we argue why we are allowed to charge it, we need to give some names. Let e be the edge of D that is not shared with A and is not flipped. Let B be the non-containing triangle that shares e and let v be the vertex that is shared by A, B and D. Now, any separating triangle that shares v and contains D must contain B as well. If it did not, it would have to share e with D, but D already shares an edge with a containing triangle and cannot share more (Lemma 7). Since the second invariant requires only a single charged free edge for each vertex of a separating triangle, it is enough that v still has an incident free edge in B.

This shows that we can charge 5 coins for every flip, while maintaining the invariants. Now all that we need to show is that after performing these flips we have indeed removed all separating triangles. As long as our triangulation has a separating triangle, we can always find a deepest separating triangle D. Since D shares at most one edge with separating triangles (Lemma 7), one of the cases above must apply to D. This gives us an edge of D to flip and five edges to charge, each of which is guaranteed by the invariants to have a coin. Therefore the process stops only after all separating triangles have been removed. $\square$

Corollary 4 *The diameter of the flip graph of all triangulations on n vertices is at most $5.2n - 24.4$.*

Proof. Mori *et al.* [4] showed that any two 4-connected triangulations can be transformed into each other by at most $4n - 22$ flips. By Theorem 3, we can make a triangulation 4-connected using at most $\lfloor (3n-6)/5 \rfloor$ flips. Hence, we can transform any triangulation into any other using at most $2 \cdot (3n-6)/5 + 4n - 22 \leq 5.2n - 24.4$ flips. $\square$

3 Lower Bound

In this section we present a lower bound on the number of flips that are required to remove all separating triangles from a triangulation. Specifically, we present a triangulation that has $(3n-10)/5$ edge-disjoint separating triangles, thereby showing that there are triangulations that require this many flips to make them 4-connected.

The triangulation that gives rise to the lower bound is constructed recursively and is similar to the Sierpiński triangle [5]. The construction starts with an empty triangle. The recursive step consists of adding an inverted triangle in the interior and connecting each vertex of the new triangle to the two vertices of the opposing edge of the original triangle. This is recursively applied to the three new triangles that share an edge with the inserted triangle, but not to the inserted triangle itself. After k iterations, instead of applying the recursive step again, we add a single vertex in the interior of each triangle we are recursing on and connect this vertex to each vertex of the triangle. We also add a single vertex in the exterior face so that the original triangle becomes separating. The resulting triangulation is called $\mathcal{T}_k$. Figure 5 illustrates this process for $\mathcal{T}_1$ and $\mathcal{T}_2$.

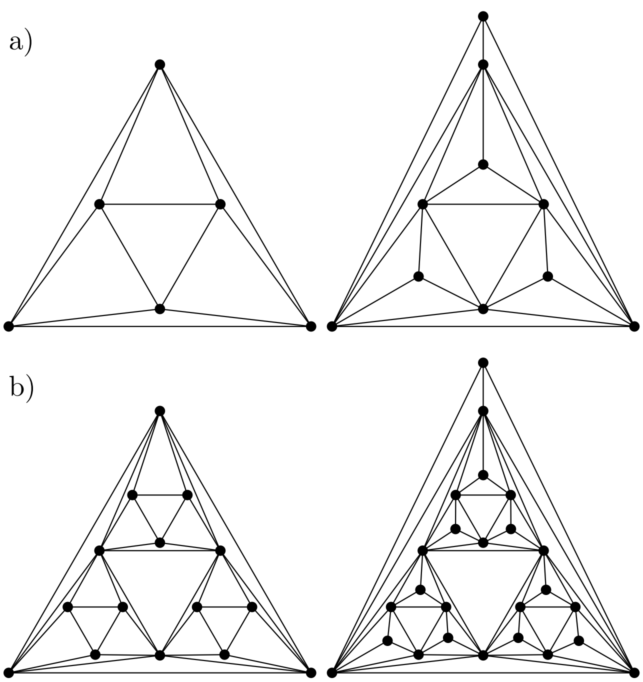

Figure 5: Triangulations $\mathcal{T}_1$ (a) and $\mathcal{T}_2$ (b), before and after the final step of the construction.

Theorem 5 *There are triangulations that require $\lceil (3n-10)/5 \rceil$ flips to make them 4-connected.*

Proof. In the construction scheme presented above, each of the triangles we recurse on becomes a separating triangle that does not share any edges with the original triangle or the other triangles that we recurse on. Thus all these separating triangles are edge-disjoint. But how many of these triangles do we get? Let L_i be the number of triangles that we recurse on after i iterations of the construction, so $L_0 = 1$, $L_1 = 3$, etc. Now let V_i be the number of vertices of $\mathcal{T}_i$. We can see that $V_1 = 10$ and if we transform $\mathcal{T}_1$ into $\mathcal{T}_2$, we have to remove each of the interior vertices added in the final step and replace them with a configuration of 6 vertices. So to get $\mathcal{T}_2$, we add 5 vertices in each of the L_1 triangles. This is true in general, giving

$$V_i \quad = \quad V_{i-1} + 5L_{i-1} \quad = \quad 10 + 5\sum_{j=2}^{i} L_{j-1} \qquad (1)$$

Let S_i be the number of separating triangles of $\mathcal{T}_i$. We can see that $S_1 = 4$ and each recursive refinement of a separating triangle leaves it intact, while adding 3 new ones. Therefore

$$S_i \quad = \quad S_{i-1} + 3L_{i-1} \quad = \quad 4 + 3\sum_{j=2}^{i} L_{j-1} \qquad (2)$$

From Equation (1), we get that

$$\sum_{j=2}^{i} L_{j-1} \quad = \quad \frac{V_i - 10}{5}$$

Substituting this into Equation (2) gives

$$S_i \quad = \quad 4 + 3\frac{V_i - 10}{5} \quad = \quad \frac{3V_i - 10}{5}$$

Since each flip removes only the separating triangle that the edge belongs to, we need $(3n-10)/5$ flips to make this triangulation 4-connected. $\square$

4 Lemmas and proofs

This section contains proofs for the technical lemmas used in the proof of Theorem 3. The proofs use the following result, which is proven in Lemmas 11 and 12 in the appendix.

Lemma 6 *A separating triangle A contains a separating triangle B if and only if there is a vertex of B inside A.*

Lemma 7 *A separating triangle can share at most one edge with containing triangles.*

Proof. Suppose we have a separating triangle D that shares two of its edges with separating triangles that contain it. First of all, these triangles cannot be the same, since then they would be forced to share the third edge as well, which means that they are D. Since a triangle does not contain itself, this is a contradiction. So call one of these triangles A and call one of the triangles that shares the other edge B. Let x, y and z be the vertices of D, such that x is shared with A and B, y is shared only with A and z is shared only with B. Let v be the vertex of B that is not shared with D.

By Lemma 6, z must be inside A, while y must be inside B, since in both cases the other two vertices of D are shared and therefore not in the interior. But this means that A contains B and B contains A. This is a contradiction, since by transitivity it would imply that the interior of A is a subgraph of itself with a strictly smaller vertex set. □

Lemma 8 *A separating triangle D that shares no edge with containing triangles can share at most one vertex with containing triangles.*

Proof. Suppose that D shares two of its vertices with containing triangles. First, both vertices cannot be shared with the same containing triangle, since then the edge between these two vertices would also be shared. Now let A be one of the containing triangles and let B be one of the containing triangles sharing the other vertex. By Lemma 6, there must be a vertex of D inside A. So then both vertices of D that are not shared with A must be inside A, otherwise there would be an edge between the interior and the exterior of A. In particular, the vertex shared by B and D lies inside A, which means that A contains B. But the reverse is also true, so B contains A as well, which is a contradiction. □

Lemma 9 *A separating triangle that shares an edge with a containing triangle cannot share the unshared vertex with another containing triangle.*

Proof. Suppose we have a separating triangle $D = (x, y, z)$ that shares an edge (x, y) with a containing triangle A and the other vertex z with another containing triangle B. By Lemma 6, x and y have to be inside B, since they cannot be outside B and they cannot be shared with B by Lemma 7. Since x and y are vertices of A, this means that B contains A. Similarly, z has to be inside A and since it is a vertex of B, A contains B. This is a contradiction. □

Lemma 10 *Given two separating triangles A and B that share an edge e, any separating triangle that contains A but not B must use e.*

Proof. Suppose that we have a separating triangle D that contains A, but not B and that does not use one of the vertices v of e. By Lemma 6, v must be inside D. But then D would also contain B, as v is a vertex of B as well. Therefore D must share both vertices of e and hence e itself. □

5 Conclusions and future work

We showed that any triangulation can be made 4-connected using at most $\lfloor(3n-6)/5\rfloor$ flips, while there are triangulations that require $\lceil(3n-10)/5\rceil$ flips. Since the difference is less than a single flip, these bounds are tight. An obvious question is how to compute the necessary flips efficiently. If we only guarantee that we use at most $n-4$ flips, it is possible to compute the set of edges to be flipped in $O(n)$ time. If we want to stay below the upper bound however, we only have an algorithm that computes the set of edges used in the proof in $O(n^2)$ time.

Another interesting problem is to minimize the number of flips to make a triangulation 4-connected. We showed that our technique is worst-case optimal, but there are cases where far fewer flips would suffice. There is a natural formulation of the problem as an instance of 3-hitting set, where the subsets correspond to separating triangles and we need to pick a minimal set of edges such that we include at least one edge from every separating triangle. This gives a simple 3-approximation algorithm that picks an arbitrary separating triangle and flips all shared edges or an arbitrary edge if there are no shared edges. However, it is not clear whether the problem is NP-hard, so it might even be possible to compute the optimal sequence in polynomial time.

Our result implies a new bound of $5.2n - 24.4$ on the diameter of the flip graph. It is likely that this can be reduced further. For example, all of the current algorithms use the same, single, canonical form. This leaves several interesting questions open. Is there another canonical form that gives a better upper bound? Can we gain something from using multiple canonical forms and picking the closest? And can we find or approximate the actual shortest flip path?

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Appendix

Lemma 11 *If a separating triangle A contains a separating triangle B, then there is a vertex of B inside A and no vertex of B can lie outside A.*

Proof. Let z be a vertex in the interior of B and let y be a vertex of A that is not shared with B. Since the interior of B is a subgraph of the interior of A and y is not inside A, y must be outside B. Since every triangulation is 3-connected, there is a path from z to y that stays inside A. This path connects the interior of B to the exterior, so there must be a vertex of B on the path and hence inside A.

Now suppose that there is another vertex of B outside A. Since all vertices of a triangle are connected by an edge, there is an edge between this vertex and the vertex of B inside A. This contradicts the fact that A is a separating triangle, so no such vertex can exist. $\square$

Lemma 12 *If a vertex x of a separating triangle B is inside a separating triangle A, then A contains B.*

Proof. Let y be a vertex of A that is not shared with B. There is a path from y to the outer face that stays in the exterior of A. There can be no vertex of B on this path, since this would create an edge between the interior and exterior of A. Therefore y is outside B.

Now suppose that A does not contain B. Then there is a vertex z inside B that is not inside A. There must be a path from z to x that stays inside B. Since x is inside A, there must be a vertex of A on this path. But since y is outside B, this would create an edge between the interior and exterior of B. Therefore A must contain B. $\square$