External-Memory Data Structures.

Basic Model:
- Data lives on an external-memory.
- Access is via <u>blocks</u> of size B.

We measure cost as the number of <u>block transfers</u>.

<u>Dictionary Problem</u>

We want to insert, delete, and search in a dictionary that stores comparable elements.

<u>Lower Bound</u>: Searching requires $\Omega(\log_{B+1} n)$ time.

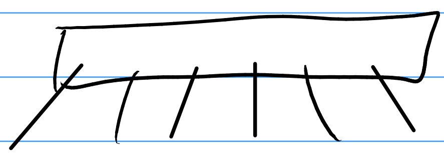
<u>Proof</u>: Reading a block only has $B+1$ possible outcomes with respect to a query value x.

$\Rightarrow$ Search algorithm gives a code whose average length is

$$(\text{average search time}) \cdot \log_2(B+1) \geq \log_2 n$$

$$\Rightarrow \text{average search time} \geq \frac{\log_2 n}{\log_2(B+1)} = \log_{B+1} n.$$

Static Structure: Use a tree where each node has $B+1$ children, and store a node in $O(1)$ blocks.



Tree has depth $\lceil \log_{B+1} n \rceil$, so following a root-to-leaf search path takes $O(\log_{B+1} n)$ block transfers.

QED.

Dynamic structure: B-tree.

- All data is stored in the leaves. (and copied in internal nodes)

- All nodes, except root, store between $\lceil B/2 \rceil$ and $2B$ items.

- Root stores at least one item

Height of tree is at most $\lceil \log_{\lceil B/2 \rceil + 1} n \rceil$, so search time is $O(\log_{B+1} n)$.

Insertion: Do a search. If leaf is not full then just store new element in leaf.

- If leaf is full then split into 2 blocks of size B and $B+1$.

- Recursively insert head of block into the parent.

Deletion: Do a search and remove element from leaf.

- If leaf contains $< B/2$ elements,
  - borrow from neighbouring leaf.
  - if neighbouring leaf has size $B/2$,
    - merge two leaves to get block of size $B-1$
    - recursively delete from parent.

Analysis: Define potential of a block as

$$\Phi(\text{block}) = \frac{c \cdot \left|\#\text{elements in block} - B\right|}{B}.$$

- Amortized cost of splitting and merging blocks is O.

- Normal case of insertion/deletion increases potential by c/B.

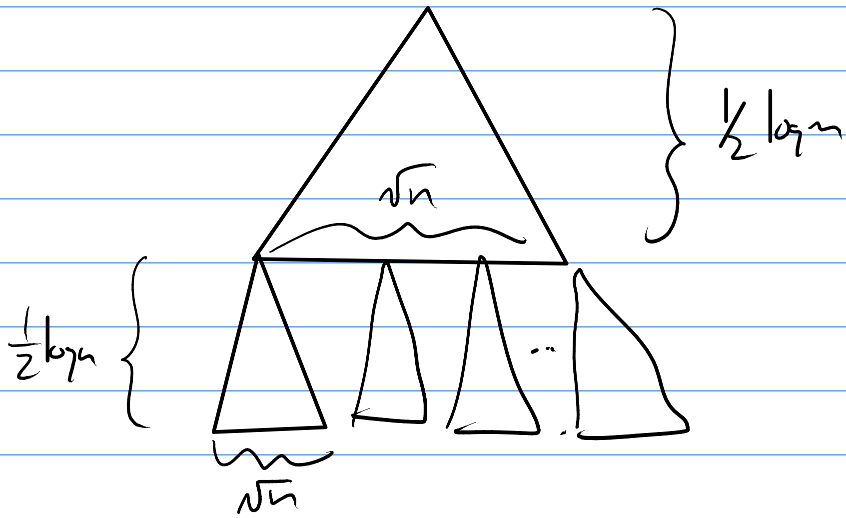
$\Rightarrow$ n insertions/deletions result in $O(\log_{B+1} n + n/B)$ block transfers.

Cache-Oblivious Model: Same as regular model, but the value of B is unknown.

- Want a data structure that is efficient for any value of B.

Static Data Structure.

- van Emde Boas layout.



and so on, recursively.

Levels of detail

- At level 0 we have one big tree. of size n
- At level 1 we have trees of size $n^{1/2}$
- At level 2 we have $\sim n^{1/4}$ trees of size $n^{1/4}$

$$\vdots$$

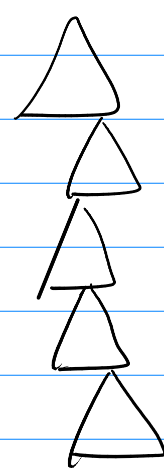
- Consider coarsest level of detail where subtree size is at most B.

  - This has subtrees of size B', where

$$B^{1/2} \leq B' \leq B$$

- These subtrees have height at least $\log B^{1/2} = \frac{1}{2} \log B$.
- Each subtree is stored contiguously in memory, so occupies at most 2 memory blocks.

- A search proceeds through $\dfrac{\log_2 n}{\frac{1}{2}\log B}$ subtrees, so it visits $O(\log_B n)$ blocks.



$\Rightarrow$ A search takes $O(\log_B n)$ time, even though we don't know the value of B.

Making it Dynamic

Packed arrays: Store n keys in an array of length $O(n)$

- Support insertion and deletion
- Maintain bounded density:
  - A subarray of length L contains $\Theta(L)$ values.

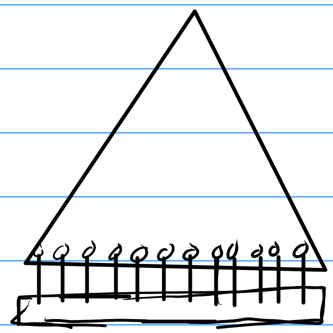
- Insertion and deletion can be done with $O(\log^2 n / B)$ block updates. (amortized).

$$\frac{\log^2 n}{B} < \log_B n \quad \text{when} \quad B \geq \log n \cdot \log\log n \quad (*)$$

E.g. $B = 1024$, $(*)$ is true for $\underbrace{n \leq 2^{128}}_{\text{more than the number of atoms in the universe.}}$

Idea: Treat the array like a binary tree that has tighter and tighter occupancy constraints on its nodes as it approaches the root. Rebuild a subtree (consecutive subarray) when one of its children violates its occupancy constraint.

Data Structure: vEB layout on top of a packed array.

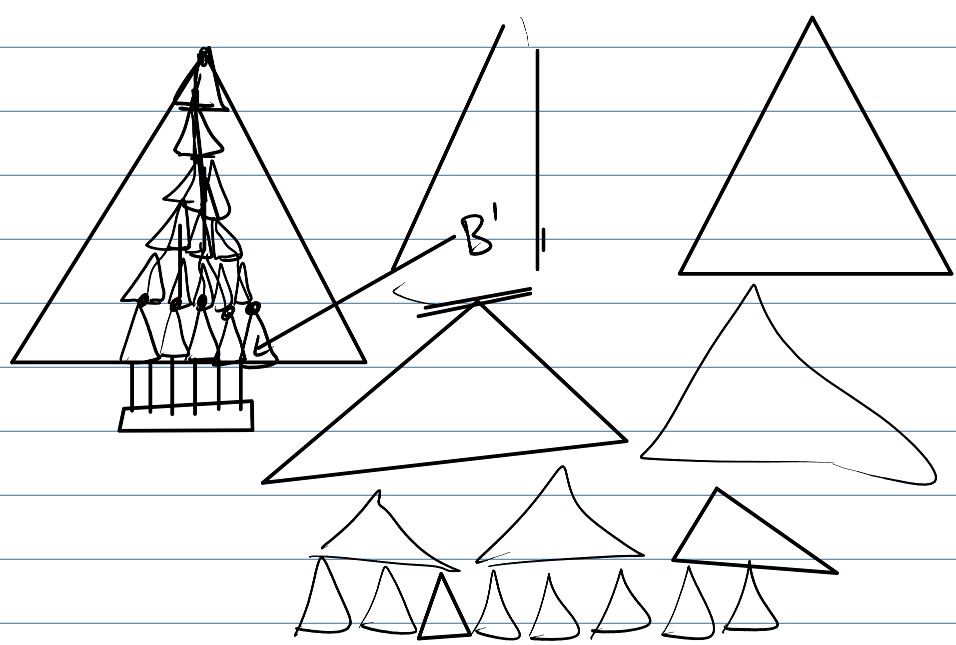


Each node in vEB tree stores the maximum of the non-empty values in its subtree.

$\Rightarrow$ Search takes $O(\log_B n)$ time.

Update requires redistributing segments of the packed array

$\Rightarrow$ fix values in the vEB tree.



- Requires $O(K/B + \log_B N)$ time.

$\Rightarrow$ This structure supports updates in

$$O\left(\frac{\log^2 n}{B} + \log_B n\right) \text{ time.}$$

Speeding-Up Updates.

To speed up updates, use indirection so that the leaves point to blocks of $\Theta(\log n)$ elements. Most insertions or deletions only operate on a block, at a cost of $O(\log n / B) \leq O(\log_B n)$.

One out of every $\Theta(\log n)$ operations operates on the real tree, at a cost of $\Theta(\log^2 n / B)$, for an amortized cost of $O(\log n / B)$ per operation.