COMP4804 SAMPLE MIDTERM EXAM

1 Finding a Large Independent Set in a 3-Regular Graph

In this question we give an algorithm to find a large independent set in a 3-regular graph. (An independent set is a set of vertices, no two of which are adjacent.)

We have a graph $G = (V, E)$ in which every vertex has degree 3 and $|V| = n$. For each vertex $v \in V$, we color v *black* with probability p and color it *white* with probability $1 - p$ and this is done indpendently for each vertex. We say that a vertex v is *good* if v is colored black and v's three neighbours are all colored white.

1. What is the probability that a particular vertex v is good?
2. What is the expected number of good vertices?
3. The above algorithm does not generate a very large independent set. It's main advantage is that it's easy to implement in parallel. Describe and analyze a simple algorithm that guarantees an independent set of size at least $n/4$. (Hint: The solution relies only on the fact that G is 3-regular.)

2 The height of a skiplist

Suppose we start with a list $L_0 = l_1, \ldots, l_n$. We obtain a new list L_1 by tossing a fair coin for each element l_i and adding l_i to L_1 iff the coin toss comes up heads.

1. What is the probability that l_i is in L_1? From this, compute the expected size of L_1.
2. Suppose we continue in this manner to obtain a list L_2 by tossing coins for each element of L_1. In general, to obtain L_i ($i > 0$), we toss a coin for each element in L_{i-1} and add that element to L_i iff the coin toss comes up heads.

   What is the probability that any particular element l_j is in L_i? From this, compute the expected size of L_i.
3. Show that the expected time required to build all the lists $L_1, L_2, L_3, \ldots$ is $O(n)$.
4. The *height* of a skiplist is the maximum value h such that L_h is non-empty. Prove that

$$\Pr\{h > \log_2 n + c\} \leq 1/2^c \quad .$$

3 Indiana's Pi

In 1897, a bill came up in the Indiana Legislature that proposed defining the value of π as $\hat{\pi} = 3.2$.

Consider the following algorithm: Generate n points uniformly in the unit square $[0,1]^2$. Count the number, k of these points whose x and y coordinates satisfy $x^2 + y^2 \leq 1$. Output the value $X = k/n$.

1. What is $\mathrm{E}[X]$?
2. What kind of random variable is k?
3. Give an upper bound on $\Pr\{k > 3.2n/4\}$.