Probability Space: (S, Pr)

- S is non-empty countable set.
- $Pr: S \to [0,1]$ such that $\sum_{\omega \in S} Pr(\omega) = 1 = Pr(S)$

• For any $A \subseteq S$ $Pr(A) = \sum_{\omega \in A} Pr(\omega)$.

• For disjoint $A, B \subseteq S$, $Pr(A \cup B) = Pr(A) + Pr(B)$. [Sum Rule].

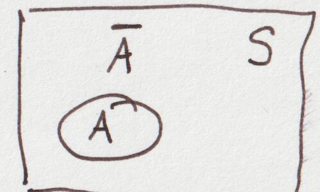
$Pr(A \cup \bar{A}) = Pr(S) = 1$.

$Pr(A \cup \bar{A}) = Pr(A) + Pr(\bar{A})$

$1 = Pr(A) + Pr(\bar{A})$.

$Pr(A) = 1 - Pr(\bar{A})$.

[Complement Rule].



Birthday Paradox.

$S = \{(b_1, b_2, b_3, \dots, b_n) : b_1 \dots b_n \in \{1, 2, 3, \dots d\}\}$. $|S| = d^n$

$Pr(\omega) = \frac{1}{|S|} = \frac{1}{d^n}$.

W = "two people have the same birthday"

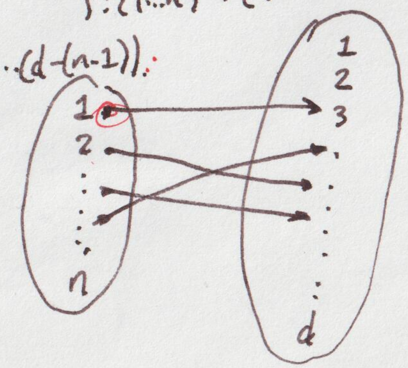
= "there exist i, j with $i \neq j$ such that $b_i = b_j$"

$\bar{W}$ = "every body has a distinct birthday". Let $f(i) = b_i$

"f is a one-to-one function"

$f: \{1 \dots n\} \to \{1 \dots d\}$.

$Pr(W) = \frac{|W|}{|S|}$ $\quad |\bar{W}| = \frac{d!}{(d-n)!} = d \cdot (d-1) \cdot (d-2) \cdots (d-(n-1))$.



n = #people.

d = # days in the year (365).

$n=2$. $S = \{(b_1, b_2) : b_1, b_2 \in \{1 \dots d\}\}$.

$|S| = d^2$.

$W = \{(b,b) : b \in \{1, \dots, d\}\}$ $|W| = d$

$Pr(W) = \frac{|W|}{|S|} = \frac{d}{d^2} = \frac{1}{d}$.

$n=3$

$W = \{(b,b,b_2) : b, b_2 \in \{1 \dots d\}\}$ d^2 W_1

$\cup \{(b, b_2, b) \quad - - - \}$ d^2 W_2

$\cup \{(b_2, b, b) \quad \}$ d^2. W_3

$|W| = 3d^2$ $\quad |W| = |W_1| + |W_2| + |W_3|$

$W = W_1 \cup W_2 \cup W_3$

$$Pr(\overline{W}) = \frac{|\overline{W}|}{|S|} = \frac{d!}{(d-n)! \cdot d^n} = \frac{d \cdot (d-1) \cdot (d-2) \cdots (d-(n-1))}{d \cdot d \cdot d \cdots d}$$

$$Pr(W) = 1 - Pr(\overline{W}) = 1 - \frac{d!}{(d-n)! \cdot d^n}$$

$d = 365$ ②

$n = 22 \quad P_{22} = 0.476 < \frac{1}{2}$

$n = 23 \quad P_{23} = 0.507 > \frac{1}{2}$.

$\binom{23}{2} \simeq 250. \quad \frac{n(n-1)}{2} \simeq 250$

$\frac{250}{365} \quad \frac{n(n-1)}{2} \simeq 365$

For x close to 0 $\quad e^x \approx 1 + x. \quad e^{-x} \approx 1 - x.$

$\frac{n}{d}$ is "close to" $\emptyset$

$1 - \frac{n}{d} \approx e^{-n/d}$.

$$Pr(\overline{W}) = 1 \cdot \left(\frac{d-1}{d}\right) \cdot \left(\frac{d-2}{d}\right) \cdot \left(\frac{d-3}{d}\right) \cdots \left(\frac{d-(n-1)}{d}\right)$$

$$= 1 \cdot \left(1 - \frac{1}{d}\right)\left(1 - \frac{2}{d}\right) \cdot \left(1 - \frac{3}{d}\right) \cdots \left(1 - \frac{n-1}{d}\right).$$

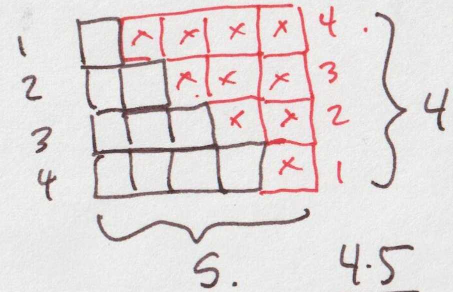
$$= e^0 \cdot e^{-\frac{1}{d}} \cdot e^{-\frac{2}{d}} \cdot e^{-\frac{3}{d}} \cdots \left(e^{-\frac{n-1}{d}}\right)$$

$$= e^{0 - \frac{1}{d} - \frac{2}{d} - \frac{3}{d} - \cdots - \frac{(n-1)}{d}}$$

$$= e^{-\frac{n(n-1)}{2d}} = e^{-\frac{n^2 - n}{2d}} = \frac{1}{e^{\frac{n^2 - n}{2d}}} = \frac{1}{e}.$$

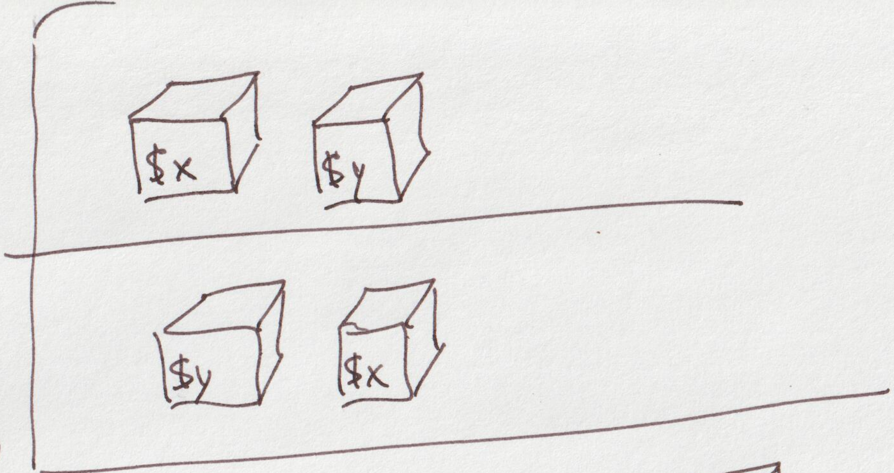
$$e^a \cdot e^b = e^{a+b}.$$

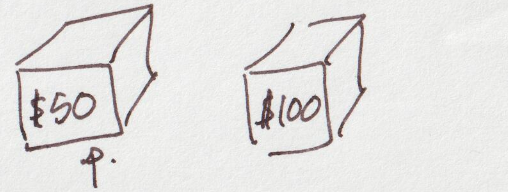
$$\underbrace{\underbrace{eeeee}_{a}\ \underbrace{eeeeeee}_{b}}_{a+b}$$



$\frac{4 \cdot 5}{2}$.

Big Box Game.





$x, y \in \{0, \dots\dots, 100\}$, $x < y$.

- Pick $z \in \{0, \dots, 99\}$ at random
- You choose a box at random, open it and count $\$a$ in the box.

  [$a \in \{x, y\}$ but you don't know x or y].

- You choose to keep $\$a$ or take the amount in the unopened box.

W = "if you get $\$y$"

~~You know $z \in \{0, \dots, 99\}$ such that $x < z + \frac{1}{2} < y$~~.

- If $a < z + \frac{1}{2}$ then switch. [$a = x$].
- If $a > z + \frac{1}{2}$ the keep a [$a = y$].

$$|W| = |W_x \cup W_y| = |W_x| + |W_y| = 100 - x + y = 100 + y - x \geq 101$$

$$Pr(W) = \frac{|W|}{|S|} \geq \frac{101}{200} > \frac{1}{2}.$$

$$\frac{100 + (y-x)}{200}$$

$S = \{(a, z) : a \in \{x, y\}, z \in \{0, 1, \dots, 99\}\}$.

$|S| = 200$.

W = "you win the game"

W_x = "you open the box containing $\$x$ and then switch"

$= \{(x, z) : z \in \{x, x+1, \dots, 99\}\}$ $|W_x| = 100 - x$

W_y = "you open the box with $\$y$ and keep it"

$= \{(y, z) : z \in \{0, 1, \dots, y-1\}\}$ $|W_y| = y$.