Assignment 2 Solutions

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2 Arrangements of MOOSONEE

1. We will use the product rule. The final string will have 8 letters.
   (a) Choose the locations of the three Os in one of $\binom{8}{3}$ ways (5 empty positions remain).
   (b) Choose the locations of the two Es in one of $\binom{5}{2}$ ways (3 empty positions remain).
   (c) Choose the location of the M in one of $\binom{3}{1}$ ways (2 empty positions remain).
   (d) Choose the location of the S in one of $\binom{2}{1}$ ways (1 empty position remains).
   (e) Place the N in the last remaining empty position in on of $\binom{1}{1}$ ways.

   Therefore, the number of distinct orderings of the letters in MOOSONEE is

$$\binom{8}{3} \times \binom{5}{2} \times \binom{3}{1} \times \binom{2}{1} \times \binom{1}{1} = 3\,360 \ .$$

3 Self-Inverting Functions

1. Let $B \subset S$ be the set of fixed points of f and let $A = S \setminus B$. Then, for every $x \in A$, $x \neq f(x)$ but $x = f(f(x))$. Therefore, the elements of A can be partitioned into two disjoint sets A_1 and A_2 such that f is a bijection from A_1 onto A_2. By the bijection rule, $|A_1| = |A_2|$. Therefore,

$$|S| - k = |A| = |A_1| + |A_2| + 2|A_1| \ .$$

   Since $|A_1|$ is an integer, $2|A_1|$ is even.

2. Let S be an n-element set and let X be the set of self-inverting functions $f : S \to S$.

   The hard part is counting the number of self-inverting functions with no fixed points, so let's count those first. The hardest part of this is avoiding double-counting (counting the same function more than once). Using the notation above, let $A \subset S$ have even size and let X_A be the set of self-inverting functions $f : A \to A$ with no fixed points. We want to determine $|X_A|$. Consider the following procedure:

(a) Choose a set $A_1 \subset A$ of $|A|/2$ elements in A and let $A_2 = A \setminus A_1$. By the definition of binomial coefficients, there are $\binom{|A|}{|A|/2}$ ways to do this.

(b) Choose a one-to-one function $f : A_1 \to A_2$. We've seen several times that there are $(|A|/2)!$ ways to do this. (For example, it's a consequence of Theorem 3.1.2 when $n = m = |A|/2$.)

(c) For each $x \in A_1$, let $y = f(x)$ and define $f(y) = x$. There is only one way to do this.

Therefore there are $\binom{|A|}{|A|/2|} \times (|A|/2)! \times 1$ ways to execute this procedure.

This procedure produces a self-inverting function $f : A \to A$ with no fixed points. In other words, it produces an element of X_A. However, for a particular $f \in X_A$, there is more than one execution of this procedure that generates f. Indeed, if f is the function defined by $f(x_i) = y_i$ and $f(y_i) = x_i$ for $i \in \{1, \ldots, |A|/2\}$, then any execution of the procedure above that, for each $i \in \{1, \ldots, |A|/2\}$

(a) item puts x_i in A_1 and y_i in A_2 and set $f(x_i) = y_i$; or

(b) puts x_i in A_2 and y_i in A_1 and sets $f(y_i) = x_i$,

will produce the function f. Therefore, there are exactly $2^{|A|/2|}$ executions of the procedure that generate f, so

$$\binom{|A|}{|A|/2} \times (|A|/2)! = 2^{|A|/2}|X_A|$$

so

$$|X_A| = \left(\frac{1}{2^{|A|/2}}\right)\binom{|A|}{|A|/2}(|A|/2)! \ .$$

Now we can easily finish up using the Product Rule and the Sum Rule. If we want a function $f : S \to S$ with exactly $2k$ fixed points, then we choose the set $B \subset S$ of $2k$ fixed points, let $A = S \setminus B$ and then choose a self-inverting function $f : A \to A$ with no fixed points. There are $\binom{n}{2k}$ ways to perform the first step and, from the preceding discussion, there are $|X_A|$ ways to perform the second step. Therefore, the number of self-inverting functions $f : S \to S$ with exactly $2k$ fixed points is

$$\binom{n}{2k}\left(\frac{1}{2^{n/2-k}}\right)\binom{n-2k}{n/2-k}(n/2-k)!$$

Finally, for each $k \in \{0, \ldots, n/2\}$, let X_k be the set of self-inverting functions $f : S \to S$ with exactly $2k$ fixed points.[1] By the Sum Rule,

$$|X| = \sum_{k=0}^{n/2} |X_k| = \sum_{k=0}^{n/2} \binom{n}{2k}\left(\frac{1}{2^{n/2-k}}\right)\binom{n-2k}{n/2-k}(n/2-k)! \ ,$$

as required.

4 Pigeonholing

1. If we look at what lossless compression means, it is that there is a compression function f and an uncompression (decompression) function g such that $g(f(x)) = x$ for any valid input x.

   In this case, the set of valid inputs, S_{1024}, of 1024-bit strings has size 2^{1024}. For any $n < 0$, the set S_n of n-bit strings has size 2^n. Therefore the set $S_{<1024}$ of bitstrings of length at most 1023 is

   $$\sum_{n=0}^{1023} |S_n| = \sum_{n=0}^{1023} 2^n = 2^{1024} - 1$$

   The set S_{1023} of 1023-bit strings has size $2^{1023} < 2^{1024}$. Therefore, by the Pigeonhole Principle, there is no one-to-one function $f : S_{1024} \to S_{<1024}$. This means that, if f is the compression function that

[1] Note that, since n is even, any self-inverting function $f : S \to S$ has an even number of fixed points.

Pied Piper claims to implement and g is the uncompression function, then there must be two different 1024-bit strings x_1 and x_2 such that $f(x_1) = y = f(x_2)$. Since the compression is lossless this means that $g(y) = x_1$ and $g(y) = x_2$. But this isn't possible, since $x_1 \neq x_2$.

2. Let $S \subseteq \{1, \dots, n\}$ have size k. Consider the set X consisting of the $\binom{k}{2}$ pairs of elements in S and let $f : X \to \{3, \dots, 2n-1\}$ be defined as $f(\{a,b\}) = a + b$. Notice that

$$|X| = \binom{k}{2} = \frac{k(k-1)}{2} \geq 2n - 1$$

since $k(k-1) \geq 4n-2$ is stated as part of the question. Therefore, by the Pigeonhole Principle f is not one-to-one (its range only has size $2n-2$), so there are two pairs $\{a,b\} \subset S$ and $\{x,y\} \subset S$ such that $f(\{a,b\}) = f(\{x,y\})$, i.e., $a+b = x+y$. Now, since $a \neq b$, $x \neq y$, $\{a,b\} \neq \{x,y\}$, and $a+b = x+y$, it must be the case that $a \neq x$, $a \neq y$, $b \neq x$, and $b \neq y$ so $\{a,b,x,y\}$ is a 4-element subset of S with $a+b = x+y$.

3. Every midpoint has an x and y coordinate that each come from the set $M = \{k/2 : k \in \{2, \dots, 2n\}$, which has size $|M| = 2n-1$. Therefore, the number of possible midpoints is at most $|M|^2 = (2n-1)^2 = 4n^2 - 2n + 1$.

Let S be a subset of G with $|S| = k$. Consider the set X consisting of the $\binom{k}{2}$ pairs of elements in S. We want to apply the Pigeonhole Principle to the midpoint function $m : X \to M^2$, so let's check:

$$\binom{k}{2} = \frac{k(k-1)}{2} > (2n-1)^2 = |M^2|$$

since $k(k+1) > 2(2n-1)^2$ is stated as part of the question. Therefore, by the Pigeonhole Principle, f is not one-to-one, so there are two pairs $\{a,b\} \in X$ and $\{x,y\} \in X$ such that $m(a,b) = m(x,y)$. Again, we can check that a, b, x, and y are all distinct, so $\{a,b,x,y\}$ is a 4-element subset of S with $m(a,b) = m(x,y)$, as required.

4. Partition Q into n^2 1×1 (unit) squares using the vertices lines $x = i$ for $i \in \{1, \dots, n-1\}$ and the horizontal lines $y = i$ for $i \in \{1, \dots, n-1\}$. The points of S are pigeons and the squares are holes. In each unit square the maximum distance between any pair of points is $\sqrt{2}$. By the Pigeonhole Principle, there are two distinct points $p, q \in S$ that are contained in the same unit square, so the distance between p and and q is at most $\sqrt{2}$, as required.

(Note: We were a bit sloppy here with the word "partition" since the n^2 unit squares overlap on their boundaries. For a point is on the boundary of 2 or more squares we can assign that point, arbitrarily, to one of those squares.)

5. Let f be the function that counts the number of zeroes in a binary string. Then $f : \{0,1\}^n \to \{0, \dots, n\}$. Thus, if S is a set of $n+2$ binary strings of length n then, by the Pigeonhole Principle $f(x) = f(y)$ for two distinct strings $x, y \in S$. So the number of zeroes in x is equal to the number of zeroes in y. But the number of ones in x and y is $n - f(x) = n - f(y)$. Therefore x and y are anagrams.

6. For any string s over the alphabet $\{a,b,c,d\}$, let s_a, s_b, s_c and s_d denote the number of a's, b's, c's and d's in s, respectively. Notice that two strings s and t are anagrams if and only if $s_a = t_a$, $s_b = t_b$, $s_c = t_c$, and $s_d = t_d$. Next, observe that, if s has length 12 then

$$s_a + s_b + s_c + s_d = 12 \ .$$

Let

$$R = \{(a,b,c,d) : a,b,c,d \in \mathbb{Z}_{\geq 0}, \quad a+b+c+d = 12\} \ .$$

We saw in class that $|R| = \binom{12+3}{3} = 455$. (This is Theorem 3.9.1 in the textbook with $n = 12$ and $k = 4$.)

Now let S be any set of 456 12-character strings over $\{a,b,c,d\}$ and let f be the function defined by $f(s) = (s_a, s_b, s_c, s_d)$, so $f : S \to R$. Since $|S| = 456 > 455 = |R|$, the Pigeonhole Principle implies that there are distinct $s, t \in S$ such that $f(s) = f(t)$, so s and t are a pair of anagrams, as required.

5 Recurrences

1. The proof is by induction on n. For the base case $n = 0$ we have

$$f(0) = 1 = 2^{0^2} \ ,$$

as required. Now assume $f(n-1) = 2^{(n-1)^2}$. Then, for $n \geq 1$,

$$\begin{aligned}
f(n) &= \frac{1}{2} \times 4^n \times f(n-1) && \text{(by definition of } f(n)\text{)} \\
&= \frac{1}{2} \times 4^n \times 2^{(n-1)^2} && \text{(by the inductive hypothesis)} \\
&= \frac{1}{2} \times 4^n \times 2^{n^2-2n+1} && \text{(since } (n-1)^2 = n^2-2n+1\text{)} \\
&= \frac{1}{2} \times 2^{2n} \times 2^{n^2-2n+1} && \text{(since } 4^n = (2^2)^n = 2^{2n}\text{)} \\
&= 2^{-1} \times 2^{2n} \times 2^{n^2-2n+1} && \text{(since } 1/2 = 2^{-1}\text{)} \\
&= 2^{n^2} \ .
\end{aligned}$$

2. To get a feel for the recurrence, we write out the first few values

n	0	1	2	3	4	5	6	7	8	9
$f(n)$	1	1	3	3	9	9	27	27	81	81

So it looks like the sequence is just powers of 3 with each power occuring twice. So $f(n) = 3^{\lfloor n/2 \rfloor}$ a natural guess and we can prove this by induction on n.

For the base cases we have $f(0) = 1 = 3^0 = 3^{\lfloor 0/2 \rfloor}$ and $f(1) = 1 = 3^0 = 3^{\lfloor 1/2 \rfloor}$, so those check out. Now assume $f(k) = 3^{\lfloor k/2 \rfloor}$ for all $k \in \{1, \ldots, n-1\}$. So,

$$\begin{aligned}
f(n) &= 3 \times f(n-2) && \text{(by definition of } f(n)\text{)} \\
&= 3 \times 3^{\lfloor (n-2)/2 \rfloor} && \text{(by the inductive hypothesis)} \\
&= 3 \times 3^{\lfloor n/2-1 \rfloor} && \text{(since } (n-2)/2 = n/2-1\text{)} \\
&= 3 \times 3^{\lfloor n/2 \rfloor - 1} && \text{(since } \lfloor x-1 \rfloor = \lfloor x \rfloor - 1\text{)} \\
&= 3^{\lfloor n/2 \rfloor}
\end{aligned}$$

as required.

3. For $n \geq 2$, any string in S_n either

   (a) begins with b followed by a string in S_{n-1};

   (b) begins with c followed by a string in S_{n-1};

   (c) begins with ab followed by a string in S_{n-2};

   (d) begins with ac followed by a string in S_{n-2}.

   Therefore, for $n \geq 2$,

$$|S_n| = 2|S_{n-1}| + 2|S_{n-2}|$$

or, in you prefer the notation we've been using, define $f(n) = |S_n|$, so we have

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2f(n-1) + 2f(n-2) & \text{if } n \geq 2 \end{cases}$$

The question gives us the solution to this recurrence, we just have to verify, using induction on n, that it's correct. Let $a = \sqrt{3}/3 + 1/2$, $b = \sqrt{3}/2 - 1/2$, $\alpha = 1 + \sqrt{3}$ and $\beta = 1 - \sqrt{3}$. We think that the solution is

$$f(n) = a\alpha^n - b\beta^n \ .$$

First we check the two base cases, starting with $n = 0$

$$\begin{aligned} a\alpha^0 - b\beta^0 &= a - b \\ &= \sqrt{3}/3 + 1/2 - \sqrt{3}/3 + 1/2 \\ &= 1 = f(0) \end{aligned}$$

and then $n = 1$

$$\begin{aligned} a\alpha^1 - b\beta^1 &= a\alpha - b\beta \\ &= (\sqrt{3}/3 + 1/2)(1 + \sqrt{3}) - (\sqrt{3}/3 - 1/2)(1 - \sqrt{3}) \\ &= (\sqrt{3}/3 + 1 + 1/2 + \sqrt{3}/2) - (\sqrt{3}/3 - 1/2 - 1 + \sqrt{3}/2) \\ &= 3 = f(1) \ . \end{aligned}$$

Now we assume that $f(k) = a\alpha^k - b\beta^k$ for all $k \in \{0, \ldots, n-1\}$. Then, for $n \geq 2$,

$$\begin{aligned} f(n) &= 2f(n-1) + 2f(n-2) && \text{(by definition)} \\ &= 2\left(a\alpha^{n-1} - b\beta^{n-1}\right) + 2\left(a\alpha^{n-2} - b\beta^{n-2}\right) \\ &= 2\left(a\alpha^{n-1} + a\alpha^{n-2}\right) - 2\left(b\beta^{n-1} - b\beta^{n-2}\right) \\ &= 2a\left(\alpha^{n-1} + \alpha^{n-2}\right) - 2b\left(\beta^{n-1} - \beta^{n-2}\right) \\ &= 2a\left(\alpha^{n-2}(\alpha + 1)\right) - 2b\left(\beta^{n-2}(\beta + 1)\right) \\ &= a\left(\alpha^{n-2}(2\alpha + 2)\right) - b\left(\beta^{n-2}(2\beta + 2)\right) \\ &= a\left(\alpha^{n-2}\alpha^2\right) - b\left(\beta^{n-2}\beta^2\right) \\ &= a\alpha^n - b\beta^n \ , \end{aligned}$$

as required.

4. Any string in S_n either
   (a) begins with a b followed by any string in S_{n-1}; or
   (b) begins with a b followed by any string in S_{n-1}; or
   (c) begins with with $k-1$ a's followed by a c followed by a string in S_{n-k} (for some $k \in \{2, \ldots, n\}$); or
   (d) consists entirely of a's.

   Therefore

$$|S_n| = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 2|S_{n-1}| + \sum_{k=2}^{n} S_{n-k-1} + 1 & \text{if } n \geq 2 \end{cases}$$

5. Here is some nave Python code to compute this sequence:

```
def f(n):
    if n == 0: return 1
    if n == 1: return 3
    return 2*f(n-1) + sum([f(n-k) for k in range(2,n+1)]) + 1

print(",".join([str(f(n)) for n in range(21)]))
```

and it produces the sequence 1,3,8,21,55,144,377,987,2584,6765,17711,46368,121393,317811,832040,2178309, 5702887,14930352,39088169,102334155,267914296. This is sequence A001906 in the OEIS (`https://oeis.org/A001906`).

6. This recurrence solves to

$$f(n,k) = \binom{n}{k}$$

We can prove this by induction on $n+k$. If $n+k=0$, then $n=k=0$ and $f(n,k)=1$ by definition and $\binom{0}{0}=1$, also by definition. When $n+k \geq 2$ then there are two cases to consider:

(a) $n > k$. In this case

$$f(n,k) = f(n-1,k) + f(n-1,k-1) = \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

where the last step is an application of Pascal's Identity.

(b) $n = k$. In this case

$$f(n,n) = f(n-1,n) + f(n-1,n-1) = 0 + \binom{n-1}{n-1} = 1 = \binom{n}{n}$$

as required.