Pathwidth and Three-Dimensional Straight-Line Grid Drawings of Graphs [*]

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Abstract. We prove that every n-vertex graph G with pathwidth $\mathtt{pw}(G)$ has a three-dimensional straight-line grid drawing with $O(\mathtt{pw}(G)^2 \cdot n)$ volume. Thus for graphs with bounded pathwidth the volume is $O(n)$, and it follows that for graphs with bounded treewidth, such as series-parallel graphs, the volume is $O(n \log^2 n)$. No better bound than $O(n^2)$ was previously known for drawings of series-parallel graphs. For planar graphs we obtain three-dimensional drawings with $O(n^2)$ volume and $O(\sqrt{n})$ aspect ratio, whereas all previous constructions with $O(n^2)$ volume have $\Theta(n)$ aspect ratio.

1   Introduction

The study of straight-line graph drawing in the plane has a long history; see [35] for a recent survey. Motivated by interesting theoretical problems and potential applications in information visualisation [33], VLSI circuit design [24] and software engineering [34], there is a growing body of research in three-dimensional straight-line graph drawing.

Throughout this paper all graphs G are undirected, simple and finite with vertex set $V(G)$ and edge set $E(G)$; $n = |V(G)|$ denotes the number of vertices of G. A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *three-dimensional drawing*, represents the vertices by distinct points in 3-space with integer coordinates (called *grid-points*), and represents each edge as a line-segment between its end-vertices, such that edges only intersect at common end-vertices. If a three-dimensional drawing is contained in an axis-aligned box with side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ three-dimensional drawing with *volume* $X \cdot Y \cdot Z$ and *aspect ratio* $\max\{X, Y, Z\} / \min\{X, Y, Z\}$. This paper considers the problem of producing a three-dimensional drawing of a given graph with small volume, and with small aspect ratio as a secondary criterion.

Related Work: In contrast to the case in the plane, every graph has a three-dimensional drawing. Such a drawing can be constructed using the 'moment

curve' algorithm in which vertex v_i, $1 \leq i \leq n$, is represented by the grid-point

$$(i, i^2, i^3) \ .$$

It is easily seen — compare with Lemma 4 to follow — that no edges cross. (Two edges *cross* if they intersect at some point other than a common end-vertex.) Cohen *et al.* [8] improved the resulting $O(n^6)$ volume bound, by proving that if p is a prime with $n < p \leq 2n$, and each vertex v_i is represented by the grid-point

$$(i, i^2 \bmod p, i^3 \bmod p)$$

then there is still no edge crossings. This construction is a generalisation of a two-dimensional technique due to Erdös [15]. Furthermore, Cohen *et al.* [8] proved that the resulting $O(n^3)$ volume bound is asymptotically optimal in the case of the complete graph K_n, and that every binary tree has a three-dimensional drawing with $O(n \log n)$ volume.

Calamoneri and Sterbini [5] proved that every 4-colourable graph has a three-dimensional drawing with $O(n^2)$ volume. Generalising this result, Pach *et al.* [28] proved that every k-colourable graph, for fixed $k \geq 2$, has a three-dimensional drawing with $O(n^2)$ volume, and that this bound is asymptotically optimal for the complete bipartite graph with equal sized bipartitions. If p is a suitably chosen prime, the main step of this algorithm represents the vertices in the ith colour class by grid-points in the set

$$\{(i, t, it) : t \equiv i^2 \pmod{p}\} \ .$$

The first linear volume bound was established by Felsner *et al.* [16], who proved that every outerplanar graph has a drawing with $O(n)$ volume. Their elegant algorithm 'wraps' a two-dimensional layered drawing around a triangular prism; see Lemma 5 for more on this method. Poranen [30] proved that series-parallel digraphs have upward three-dimensional drawings with $O(n^3)$ volume, and that this bound can be improved to $O(n^2)$ and $O(n)$ in certain special cases. Recently di Giacomo *et al.* [11] proved that series-parallel graphs with maximum degree three have three-dimensional drawings with linear volume.

Note that three-dimensional drawings with the vertices having real coordinates have been studied by Bruß and Frick [4], Chilakamarri *et al.* [6], Chrobak *et al.* [7], Cruz and Twarog [9], Eades and Garvan [14], Garg *et al.* [17], Hong [20], Hong and Eades [21, 22], Hong *et al.* [23], Monien *et al.* [25], and Ostry [27]. Aesthetic criteria besides volume which have been considered include symmetry [20–23], aspect ratio [7, 17], angular resolution [7, 17], edge-separation [7, 17], and convexity [6, 7, 14].

Tree Decompositions: Before stating our results we recall some definitions. A *tree decomposition* of a graph G is a tree T together with a collection of subsets T_x (called *bags*) of $V(G)$ indexed by the vertices of T such that:

– $\displaystyle\bigcup_{x \in V(T)} T_x = V(G)$,

– for every edge $vw \in E(G)$, there is a vertex $x \in V(T)$ such that the bag T_x contains both v and w, and

– for all vertices $x, y, z \in V(T)$, if y is on the path from x to z in T, then $T_x \cap T_z \subseteq T_y$.

The *width* of a tree decomposition is the maximum cardinality of a bag minus one. A *path decomposition* is a tree decomposition where the tree T is a path $T = (x_1, x_2, \ldots, x_m)$, which is simply identified by the sequence of bags $T_1, T_2, \ldots, T_m$ where each $T_i = T_{x_i}$. The *pathwidth* (respectively, *treewidth*) of a graph G, denoted by $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$), is the minimum width of a path (tree) decomposition of G. A graph G is said to have *bounded pathwidth* (*treewidth*) if $\mathsf{pw}(G) = k$ ($\mathsf{tw}(G) = k$) for some constant k. Given a graph with bounded pathwidth (treewidth), the algorithm of Bodlaender [1] determines a path (tree) decomposition with width $\mathsf{pw}(G)$ ($\mathsf{tw}(G)$) in linear time. Note that the relationship between graph drawing and pathwidth or treewidth has been previously investigated by Dujmović *et al.* [13] and Peng [29].

Our Results: Our main result is the following.

Theorem 1. *Every n-vertex graph G has an $O(\mathsf{pw}(G)) \times O(\mathsf{pw}(G)) \times O(n)$ three-dimensional drawing.*

Since $\mathsf{pw}(G) < n$, Theorem 1 matches the $O(n^3)$ volume bound discussed above; in fact, the drawings of K_n produced by our algorithm are identical to those produced by Cohen *et al.* [8]. We have the following corollary since every graph G has $\mathsf{pw}(G) \in O(\mathsf{tw}(G) \cdot \log n)$ [2].

Corollary 1. (a) *Every n-vertex graph with bounded pathwidth has a three-dimensional drawing with $O(n)$ volume.* (b) *Every n-vertex graph with bounded treewidth has a three-dimensional drawing with $O(n \log^2 n)$ volume.* □

While the notion of bounded treewidth may appear to be a purely theoretic construct, graphs arising in many applications of graph drawing do have small treewidth. For example, outerplanar graphs, series-parallel graphs and Halin graphs respectively have treewidth 2, 2 and 3 (see [2, 12]). Thus Corollary 1(b) implies that these graphs have three-dimensional drawings with $O(n \log^2 n)$ volume. While linear volume is possible for outerplanar graphs [16], our result is the first known sub-quadratic volume bound for all series-parallel and Halin graphs. Another example arises in software engineering applications. Thorup [32] proved that the control-flow graphs of go-to free programs in many programming languages have treewidth bounded by a small constant; in particular, 3 for Pascal and 6 for C. Other families of graphs having bounded treewidth (for constant k) include: almost trees with parameter k, graphs with a feedback vertex set of size k, partial k-trees, bandwidth k graphs, cutwidth k graphs, planar graphs of radius k, and k-outerplanar graphs. If the size of a maximum clique is a constant k then chordal, interval and circular arc graphs also have bounded treewidth. Thus Corollary 1(b) pertains to such graphs.

Since a planar graph is 4-colourable, by the results of Calamoneri and Sterbini [5] and Pach *et al.* [28] discussed above, every planar graph has a three-dimensional drawing with $O(n^2)$ volume. Of course this result also follows from the classical algorithms of de Fray4sseix *et al.* [10] and Schnyder [31] for producing plane grid drawings. All of these methods produce $O(1) \times O(n) \times O(n)$ drawings, which have $\Theta(n)$ aspect ratio. Since every planar graph G has $\mathsf{pw}(G) \in O(\sqrt{n})$ [2] we have the following corollary of Theorem 1.

Corollary 2. *Every n-vertex planar graph has an $O(\sqrt{n}) \times O(\sqrt{n}) \times O(n)$ three-dimensional drawing with $\Theta(\sqrt{n})$ aspect ratio.*   □

This result matches the above $O(n^2)$ volume bounds with an improvement in the aspect ratio by a factor of $\Theta(\sqrt{n})$. Our final result examines the trade-off between aspect ratio and volume.

Theorem 2. *Let G be an n-vertex graph. For every r, $1 \leq r \leq n/(\mathsf{pw}(G) + 1)$, G has a three-dimensional drawing with $O(n^3/r^2)$ volume and aspect ratio $2r$.*

2   Proofs

We first introduce a combinatorial structure which is the basis for a two-dimensional layered graph drawing. An *ordered k-layering* of a graph G consists of a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ into *layers*, and a total ordering $<_i$ of each V_i, such that for every edge vw, if $v <_i w$ then there is no vertex x with $v <_i x <_i w$. The *span* of an edge vw is $|i - j|$ if $v \in V_i$ and $w \in V_j$. An *intralayer* edge is an edge with zero span. An *X-crossing* consists of two edges vw and xy such that for distinct layers i and j, $v <_i x$ and $y <_j w$. The next lemma highlights the intrinsic relationship between three-dimensional drawings and ordered layerings.

Lemma 1. *Let G be an n-vertex graph with an $A \times B \times C$ three-dimensional drawing. Then G has an ordered AB-layering with no X-crossing, and G has an ordered $2AB$-layering with no X-crossing and no intralayer edges.*

Proof. Let $V_{x,y}$ be the set of vertices of G with an X-coordinate of x and a Y-coordinate of y, where without loss of generality $1 \leq x \leq A$ and $1 \leq y \leq Y$. Consider each set $V_{x,y}$ to be ordered $V_{x,y} = (v_{x,y,1}, \ldots, v_{x,y,n_{x,y}})$ by the Z-coordinates of its elements. Then the ordered layering $\{V_{x,y} : 1 \leq x \leq A, 1 \leq y \leq Y\}$ has no X-crossing as otherwise there would be a crossing in the original drawing. Now, define $V'_{x,y} = \{v_{x,y,j} : j \text{ odd}\}$ and $V''_{x,y} = \{v_{x,y,j} : j \text{ even}\}$, and consider these sets to be ordered as in $V_{x,y}$. Then, as in the above, the ordered layering $\{V'_{x,y}, V''_{x,y} : 1 \leq x \leq A, 1 \leq y \leq B\}$ has no X-crossing. Moreover there is no intralayer edges, as otherwise an edge between two vertices in $V'_{x,y}$ would have passed through a vertex in $V''_{x,y}$ (or vice versa) in the original drawing.   □

The proofs of Theorems 1 and 2 proceed in three steps. First, an ordered layering with no X-crossing is constructed from a given path decomposition. The

second step balances the number of vertices on each layer. The third step, which is essentially the converse of Lemma 1, takes an ordered layering with no X-crossing and assigns coordinates to the vertices to avoid edge crossings. The style of three-dimensional drawing produced by our algorithm, where vertices on a single layer are positioned on vertical 'rods', is illustrated in Fig. 1.
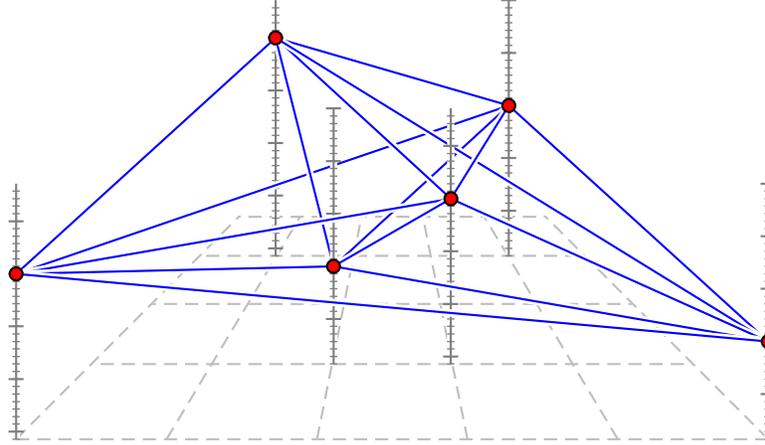

Fig. 1. A three-dimensional drawing of K_6.

Our algorithm for constructing an ordered layering makes use of the so-called normalised path decompositions of Gupta *et al.* [19]. (The more general notion of normalised tree decompositions was developed earlier by Gupta and Nishimura [18].) A path decomposition $T_1, T_2, \ldots, T_m$ of width k is *normalised* if $|T_i| = k + 1$ for all odd i and $|T_i| = k$ for all even i, and $T_{i-1} \cap T_{i+1} = T_i$ for all even i. The algorithm of Gupta *et al.* [19] normalises a path decomposition while maintaining the width in linear time.

Lemma 2. *If a graph G has a normalised path decomposition $T_1, T_2, \ldots, T_m$ of width $k - 1$, then G has an ordered k-layering with no X-crossing (see Fig. 2).*

Proof. For every vertex $v \in V(G)$, let $T_{\alpha(v)}$ and $T_{\beta(v)}$ be the first and last bags containing v. Construct an ordered k-layering of G as follows. Let $T_1 = \{v_1, v_2, \ldots, v_k\}$, and position each v_i as the leftmost vertex on layer i, $1 \leq i \leq k$. Since the path decomposition is normalised, for all bags T_j with j even, there is a unique vertex $x_j \in T_{j-1} \setminus T_j$; that is, $\beta(x_j) = j - 1$. Similarly, for all bags T_j with $j > 1$ odd, there is a unique vertex $y_j \in T_j \setminus T_{j-1}$; that is, $\alpha(y_j) = j$.

The remainder of the ordered layering is constructed by sweeping through the bags of the path decomposition as follows. For all odd $j = 3, 5, \ldots, m$, position y_j in the same layer as the vertex x_{j-1} and immediately to the right of x_{j-1}. Clearly, x_{j-1} was the rightmost vertex in the layer before inserting y_j.

Since $j - 1 = \beta(x_{j-1}) < \alpha(y_j) = j$, there is no bag containing both x_{j-1} and y_j, and no edge $x_{j-1}y_j \in E(G)$. In general, two vertices in the same layer are not in a common bag and are not adjacent.

Suppose there is an X-crossing between edges vw and xy. Without loss of generality, $v <_i x$ and $y <_j w$ for some layers i and j. Thus $\beta(v) < \alpha(x)$ and $\beta(y) < \alpha(w)$. Since vw is an edge, v and w appear in some bag together; that is, $\alpha(w) \le \beta(v)$, which implies that $\beta(y) < \alpha(x)$. This is the desired contradiction since x and y appear in some bag together. □
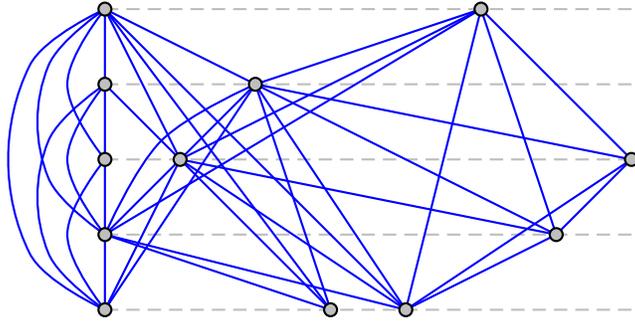

Fig. 2. An ordered 5-layering with no X-crossing produced by Lemma 2.

The second step of our algorithm is based on the algorithm of Pach *et al.* [28] for balancing the size of the colour classes in a vertex-colouring. Note that while Lemma 2 produces an ordered layering with no intralayer edges, the remaining steps of our algorithm are valid in the more general situation that the given ordered layering possibly has intralayer edges.

Lemma 3. *If a graph G has an ordered k-layering with no X-crossing, then for every $l > 0$, G has an ordered $\lfloor l + k \rfloor$-layering with no X-crossing and at most $\lceil \frac{n}{l} \rceil$ vertices in each layer.*

Proof. For each layer with $q > \lceil \frac{n}{l} \rceil$ vertices, replace it by $\lceil q/\lceil \frac{n}{l} \rceil \rceil$ 'sub-layers' each with exactly $\lceil \frac{n}{l} \rceil$ vertices except for at most one sub-layer with $q \bmod \lceil \frac{n}{l} \rceil$ vertices, such that the vertices in each sub-layer are consecutive in the original layer and the original order is maintained. There is no X-crossing between sub-layers of the same original layer as there is at most one edge between such sub-layers. There is no X-crossing between sub-layers from different original layers as otherwise there would be an X-crossing in the original layering. There are at most $\lfloor l \rfloor$ layers with $\lceil \frac{n}{l} \rceil$ vertices. Since there are at most k layers with less than $\lceil \frac{n}{l} \rceil$ vertices, one for each of the original layers, there is a total of at most $\lfloor l + k \rfloor$ layers. □

The third step of our algorithm is inspired by the generalisations of the moment curve algorithm by Cohen *et al.* [8] and Pach *et al.* [28]. Loosely speaking, Cohen *et al.* [8] allow three 'free' dimensions, whereas Pach *et al.* [28] use the assignment of vertices to colour classes to 'fix' one dimension with two dimensions free. We use an assignment of vertices to layers in an ordered layering without X-crossings to fix two dimensions with one dimension free.

Lemma 4. *If a graph G has an ordered k-layering $\{(V_i, <_i) : 1 \le i \le k\}$ with no X-crossing then G has a $k \times 2k \times 2k \cdot n'$ three-dimensional drawing, where n' is the maximum number of vertices in a layer.*

Proof. Let p be the smallest prime such that $p > k$. Then $p \le 2k$ by Bertrand's postulate. For each i, $1 \le i \le k$, represent the vertices in V_i by the grid-points

$$\{(i, i^2 \bmod p, t) : 1 \le t \le p \cdot |V_i|, t \equiv i^3 \pmod{p}\} \ ,$$

such that the Z-coordinates respect the given linear ordering $<_i$. Draw each edge as a line-segment between its end-vertices. Suppose two edges e and e' cross such that their end-vertices are at distinct points $(i_\alpha, i_\alpha^2 \bmod p, t_\alpha)$, $1 \le \alpha \le 4$. Then these points are coplanar, and if M is the matrix

$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \bmod p & t_1 \\ 1 & i_2 & i_2^2 \bmod p & t_2 \\ 1 & i_3 & i_3^2 \bmod p & t_3 \\ 1 & i_4 & i_4^2 \bmod p & t_4 \end{pmatrix}$$

then the determinant $\det(M) = 0$. We proceed by considering the number of distinct layers $N = |\{i_1, i_2, i_3, i_4\}|$.

- $N = 1$: By the definition of an ordered layering e and e' do not cross.
- $N = 2$: If either edge is intralayer then e and e' do not cross. Otherwise neither edge is intralayer, and since there are no X-crossings in the ordered layering, e and e' do not cross.
- $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $\det(M) = (t_2 - t_1) \cdot \det(M')$, where

$$M' = \begin{pmatrix} 1 & i_2 & i_2^2 \bmod p \\ 1 & i_3 & i_3^2 \bmod p \\ 1 & i_4 & i_4^2 \bmod p \end{pmatrix} \ .$$

Since $t_1 \ne t_2$, $\det(M') = 0$. However, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is non-zero since i_2, i_3 and i_4 are distinct and p is a prime, a contradiction.

- $N = 4$: Let M' be the matrix obtained from M by taking each entry modulo p. Then $\det(M') = 0$. Since $t_\alpha \equiv i_\alpha^3 \pmod{p}$, $1 \le \alpha \le 4$,

$$M' \equiv \begin{pmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{pmatrix} \pmod{p} \ .$$

Since each $i_\alpha < p$, M' is a Vandermonde matrix modulo p, and thus

$$\det(M') \equiv (i_1 - i_2)(i_1 - i_3)(i_1 - i_4)(i_2 - i_3)(i_2 - i_4)(i_3 - i_4) \pmod{p},$$

which is non-zero since $i_\alpha \neq i_\beta$ and p is a prime. This contradiction proves there are no edge crossings. The produced drawing is at most $k \times 2k \times 2k \cdot n'$.   □

We now prove the theorems.

Proof of Theorem 1. By Lemma 2, G has an ordered k-layering with no X-crossing, where $k = \mathsf{pw}(G) + 1$. By Lemma 3 with $l = k$, G has an ordered $(2k)$-layering with no X-crossing and at most $\lceil \frac{n}{k} \rceil$ vertices on each layer. By Lemma 4, G has a $2k \times 4k \times 4k \cdot \lceil \frac{n}{k} \rceil$ three-dimensional drawing, which is at most $2(\mathsf{pw}(G) + 1) \times 4(\mathsf{pw}(G) + 1) \times 4(n + \mathsf{pw}(G) + 1)$. The result follows since $1 \leq \mathsf{pw}(G) < n$.   □

Proof of Theorem 2. By Lemma 2, G has an ordered k-layering with no X-crossing, where $k = \mathsf{pw}(G) + 1$. By Lemma 3 with $l = \frac{n}{r}$, G has an ordered $\lfloor \frac{n}{r} + k \rfloor$-layering with no X-crossing and at most r vertices in each layer. By assumption $r \leq n/(\mathsf{pw}(G) + 1)$. Thus $k \leq \frac{n}{r}$ and the number of layers is at most $\frac{2n}{r}$. By Lemma 4, G has a $\frac{2n}{r} \times \frac{4n}{r} \times 4n$ three-dimensional drawing, which has volume $32n^3/r^2$ and aspect ratio $2r$.   □

3   Commentary

The following are some open problems concerning straight-line grid drawings.

1. The optimal area of a two-dimensional straight-line grid drawing of a planar graph is $\Theta(n^2)$ [10]. Can this bound be improved for two-dimensional drawings of planar graphs with bounded pathwidth or treewidth?
2. A graph with degree bounded by some constant k is $(k + 1)$-colourable, and thus by the theorem of Pach *et al.* [28], has a three-dimensional drawing with $O(n^2)$ volume. Pach *et al.* [28] ask whether every graph with bounded degree has a three-dimensional drawing with $o(n^2)$ volume?
3. As discussed in Section 1 every planar graph has a three-dimensional drawing with $O(n^2)$ volume. Felsner *et al.* [16] ask whether every planar graph has a three-dimensional drawing with $O(n)$ volume? Even a volume bound of $o(n^2)$ would be interesting.

As a final observation, we show that a generalisation of the 'wrapping' algorithm of Felsner *et al.* [16] can be applied in conjunction with our algorithm, which may be helpful in solving problems 2 and 3 above. Note that Felsner *et al.* [16] prove the case $s = 1$ (with improved constants in the volume).

Lemma 5. *Let a graph G have an ordered k-layering $\{(V_i, <_i) : 1 \leq i \leq k\}$ with no X-crossing. If the maximum edge span is s, then G has an $O(s) \times O(s) \times O(n)$ three-dimensional drawing.*

Proof. Let $t = 2s + 1$. Construct an ordered t-layering of G by merging the layers $\{V_i : i \equiv j \pmod{t}\}$ for each j, $0 \leq j \leq t - 1$, with vertices in V_α appearing before vertices in V_β in the new layer j, for all $\alpha, \beta \equiv j \pmod{t}$ with $\alpha < \beta$. The given ordering of each V_i is preserved in the new layers. It remains to prove that there is no X-crossing. Consider two edges vw and xy. Let i_1 and i_2, $1 \leq i_1 < i_2 \leq k$, be the minimum and maximum layers containing v, w, x or y in the ordered k-layering.

Firstly consider the case that $i_2 - i_1 > 2s$. Then without loss of generality v is on layer i_2 and y is on layer i_1. Thus w is on a greater layer than x, and even if x (or y) appear on the same layer as v (or w) in the new t-layering, x (or y) will be to the left of v (or w). Thus these edges do not form an X-crossing in the ordered t-layering. Otherwise $i_2 - i_1 \leq 2s$. Thus any two of v, w, x or y will appear on the same layer in the t-layering if and only if they are on the same layer in the given ordered k-layering (since $t > 2s$). Hence the only way for these four vertices to appear on exactly two layers in the ordered t-layering is if they were on exactly two layers in the given k-layering, in which case, by assumption vw and xy do not form an X-crossing.

Therefore there are no X-crossings. By Lemma 3 with $l = t$, G has an ordered $2t$-layering with no X-crossing and at most $\lceil \frac{n}{t} \rceil$ vertices in each layer. Since $t = 2s + 1$, by Lemma 4, G has a $2(2s + 1) \times 4(2s + 1) \times 4(2s + 1)\lceil \frac{n}{2s+1} \rceil$ three-dimensional drawing, which is $2(2s + 1) \times 4(2s + 1) \times 4(n + 2s)$. The result follows since $s \leq n$. □

Lemma 5 also shows that small pathwidth is not necessary for a graph to have a three-dimensional drawing with small volume. The $\sqrt{n} \times \sqrt{n}$ plane grid graph has pathwidth $\Theta(\sqrt{n})$, but has an ordered layering with maximum edge span 1. Therefore it has a three-dimensional drawing with $O(n)$ volume by Lemma 5.

Acknowledgements

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