Question 1: In class, we showed that CIRCUITSAT is **NP**-complete, by proving that $L \leq_P$ CIRCUITSAT for every language L in **NP**. Recall that

$$\text{INDEPENDENTSET} := \{(G, K) : \text{ the graph } G \text{ has an independent set of size } K\}.$$

In this question, you have to show the reduction from INDEPENDENTSET to CIRCUITSAT for a very small graph:

Let G be the graph with vertex set $\{u, v, w\}$ and edge set $\{\{u, v\}, \{v, w\}\}$ and let $K = 2$. Construct a Boolean circuit B such that

$$(G, 2) \in \text{INDEPENDENTSET if and only if } B \in \text{CIRCUITSAT}.$$

(In other words, B is the circuit corresponding to the verification algorithm for the fixed graph G.) The circuit B should have:

- Three known input gates that encode the three possible edges. The first of these gates has label TRUE, because it indicates the presence of the edge $\{u, v\}$. The second of these gates has label FALSE, because it indicates that $\{u, w\}$ is not an edge. The third of these gates has label TRUE, because it indicates the presence of the edge $\{v, w\}$.

- Three unknown input gates that encode the independent set. The first of these gates indicates whether or not u is in the independent set; the second indicates whether or not v is in the independent set; the third indicates whether or not w is in the independent set.

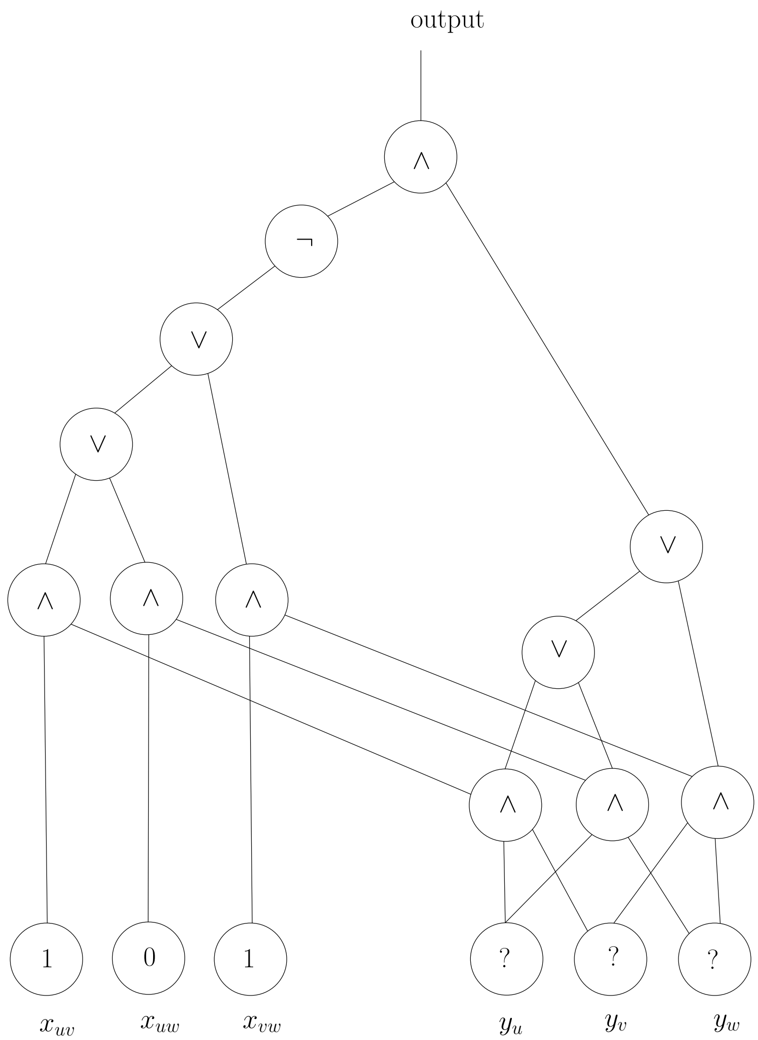
Solution: We have to come up with a Boolean circuit having three unknown input gates y_u, y_v, and y_w, such that the following is true: For any Boolean values for y_u, y_v, and y_w, the output of the circuit is TRUE if and only if y_u, y_v, and y_w encode an independent set of size at least two.

The Boolean circuit is given in the figure below. This circuit does the following:

- The output of the top $\vee$-gate on the right is TRUE if and only if at least two of y_u, y_v, and y_w are TRUE. This guarantees that we select a subset of the vertices of size at least two.

- The output of the top $\vee$-gate on the left is TRUE if and only if both vertices of some edge occur among the selected vertices. Thus, the output of this gate is FALSE if and only if the selected vertices really form an independent set (which is what we want). Therefore, the $\neg$-gate guarantees that the selected vertices form an independent set.

- The output of the Boolean circuit is TRUE if and only if the results of the previous two items are true. This is guaranteed by the $\wedge$-gate on top.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