Bellman - Ford Algorithm

1958 1956

Input: directed graph $G=(V,E)$

each edge (u,v) in E has a weight $wt(u,v)$,

weights can be negative,

however: G does not contain a directed negative-weight cycle

source vertex s.

Output: for each vertex v:

$\delta(s,v)$ = length of a shortest path from s to v.

Exercise: show that $\delta(s,s)=0$.

BF 2

As before: for each ~~xxxxx~~ vertex v, maintain a variable

$d(v)$ = length of a shortest path from s to v found so far.

<u>Bellman-Ford</u>:

for each $v \in V$: $d(v) = \infty$;

$d(s) = 0$;

for $i = 1, 2, \ldots, |V|-1$:

  for each edge (u,v) in E:

    $d(v) = \min(d(v), d(u) + wt(u,v))$

Running time: $O(|V| \cdot |E|)$

Correctness:

Claim 1: At any moment:

$$\delta(s,v) \le d(v) \text{ for every vertex } v.$$

Proof: as on page 93. $\square$

Claim 2: Assume, at some moment, $d(v)$ becomes equal to $\delta(s,v)$. Then during the rest of the algorithm, $d(v)$ does not change.

Proof: as on page 93. $\square$

Claim: For every vertex v, at the end of the for-loop, $d(v) = \delta(s,v)$.

Proof: Consider a shortest path from s to v:

$$s = v_0 \to v_1 \to v_2 \to \cdots \to v_{k-1} \to v_k = v$$

Note: this path has k edges, and $k+1$ vertices, $0 \le k \le |V|-1$. [do you see why?]

We will show: after k iterations of the outer for-loop, $d(v) = \delta(s,v)$. $\Big\}$ (*)

This, together with Claim 2 on page BF3 will imply the claim.

induction on k.

base case: $k=0$, i.e., $v=s$.

After 0 iterations of the for-loop, $d(s) = 0 = \delta(s,s)$.

$\therefore$ (*) is true

Let $k \geq 1$ and assume $\circledast$ is true for $k-1$. BF5

$s = v_0 \to v_1 \to v_2 \to \dots \to v_{k-1}$ is a shortest path from s to v_{k-1}, it has $k-1$ edges.

From induction hypothesis:

after $k-1$ iterations of the for-loop:

$$d(v_{k-1}) = \delta(s, v_{k-1}).$$

During iteration k, the algorithm considers all edges in E, in particular (v_{k-1}, v_k).

At the end of iteration k:

$$\begin{aligned} d(v_k) &\leq d(v_{k-1}) + wt(v_{k-1}, v_k) \\ &= \delta(s, v_{k-1}) + wt(v_{k-1}, v_k) \\ &= \delta(s, v_k) \\ &\leq d(v_k). \end{aligned}$$

Same reasoning as on page 103

$\therefore d(v) = \delta(s, v)$. $\square$