

Bellman - Ford Algorithm

(BF1)

↓
1958

↓
1956

Input: directed graph $G=(V,E)$

each edge (u,v) in E has a weight $wt(u,v)$,

weights can be negative,

however: G does not contain a directed
negative-weight cycle

Source vertex s .

Output: for each vertex v :

$\delta(s,v)$ = length of a shortest path
from s to v .

Exercise: show that $\delta(s,s)=0$.

As before : for each ~~vertex~~ vertex v ,
maintain a variable

BF2

$d(v)$ = length of a shortest path from s
to v found so far.

Bellman-Ford :

for each $v \in V$: $d(v) = \infty$;

$d(s) = 0$;

for $i = 1, 2, \dots, |V| - 1$:

for each edge (u, v) in E :

$d(v) = \min(d(v), d(u) + wt(u, v))$

Running time : $O(|V| \cdot |E|)$

Correctness:

BF3

Claim 1: At any moment:

$$\delta(s, v) \leq d(v) \text{ for every vertex } v.$$

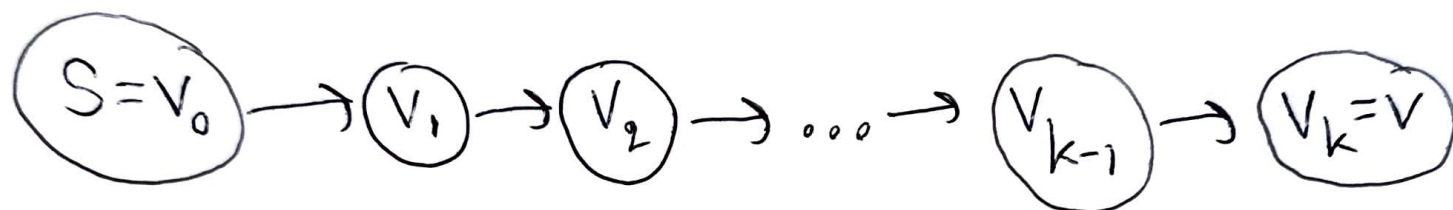
Proof: as on page 93. $\square$

Claim 2: Assume, at some moment, $d(v)$ becomes equal to $\delta(s, v)$. Then during the rest of the algorithm, $d(v)$ does not change.

Proof: as on page 93. $\square$

Claim: For every vertex v , at the end of the for-loop, $d(v) = \delta(s, v)$. BF4

Proof: Consider a shortest path from s to v :



Note: this path has k edges, and $k+1$ vertices,
 $0 \leq k \leq |V|-1$. [do you see why?]

We will show: after k iterations of the outer for-loop, $d(v) = \delta(s, v)$. ⊗

This, together ^{with} Claim 2 on page BF3 will imply the claim.

induction on k .

base case: $k=0$, i.e., $v=s$.

After 0 iterations of the for-loop, $d(s) = 0 = \delta(s, s)$.

∴ ⊗ is true

Let $k \geq 1$ and assume $(*)$ is true for $k-1$. BF5

$S = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{k-1}$ is a shortest path from s to v_{k-1} , it has $k-1$ edges.

From induction hypothesis:

after $k-1$ iterations of the for-loop:

$$d(v_{k-1}) = \delta(s, v_{k-1}).$$

During iteration k , the algorithm considers all edges in E , in particular (v_{k-1}, v_k) .

At the end of iteration k :

$$\begin{aligned} d(v_k) &\leq d(v_{k-1}) + \text{wt}(v_{k-1}, v_k) \\ &= \delta(s, v_{k-1}) + \text{wt}(v_{k-1}, v_k) \\ &= \delta(s, v_k) \\ &\leq d(v_k). \end{aligned}$$

$$\therefore d(v) = \delta(s, v).$$

□

Same
reasoning
as on
page
103