C1

from page 221: 3SAT is NP-complete.

We will show: 3Color is NP-complete.

It is clear that 3Color $\in$ NP:

certificate = color for each vertex

from page 199: 3Color $\leq_P$ 3SAT

this does not imply that 3Color is NP-complete.

We will show: 3SAT $\leq_P$ 3Color

We need function f:

1. f: input φ for 3SAT $\rightarrow$ input $f(\varphi) = G$ for 3Color
2. φ satisfiable $\Leftrightarrow$ G 3-colorable
3. time to compute G is polynomial in $|\varphi|$

Boolean variables $x_1, x_2, \ldots, x_n$

$\varphi = C_1 \wedge C_2 \wedge \ldots \wedge C_m$

each clause $C_j = \ell_{j1} \vee \ell_{j2} \vee \ell_{j3}$

literal : x_i or $\neg x_i$

We define a graph G such that

φ is satisfiable $\iff$ G is 3-colorable

we use red, green, blue

Approach : G contains $2n$ vertices

x_i , $\neg x_i$, $i = 1, 2, \ldots, n$

(there will be more vertices)

such that

(1) in any 3-coloring of G:

for $i = 1, \ldots, n$: one of x_i and $\neg x_i$ is green, the other is red

the green vertex will correspond to C3
the truth value of x_i :

x_i green $\longleftrightarrow$ $x_i = \text{true}$

$\neg x_i$ green $\longleftrightarrow$ $x_i = \text{false}$

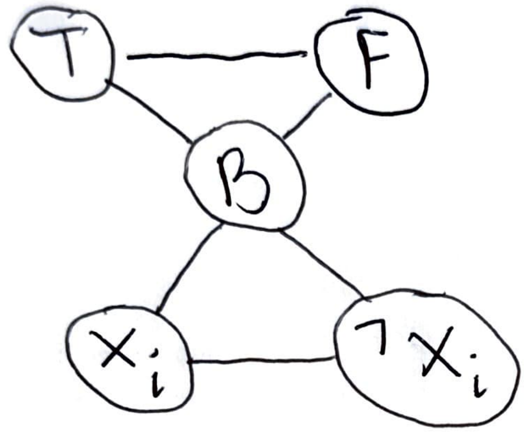
(2) for every clause $C_j = a \vee b \vee c$
(e.g. $C_j = x_1 \vee \neg x_2 \vee x_3$)

$\underbrace{C_j = \text{true}}_{\substack{\updownarrow \\ \text{at least one literal is true}}} \longleftrightarrow$ at least one of the vertices a, b, c is green

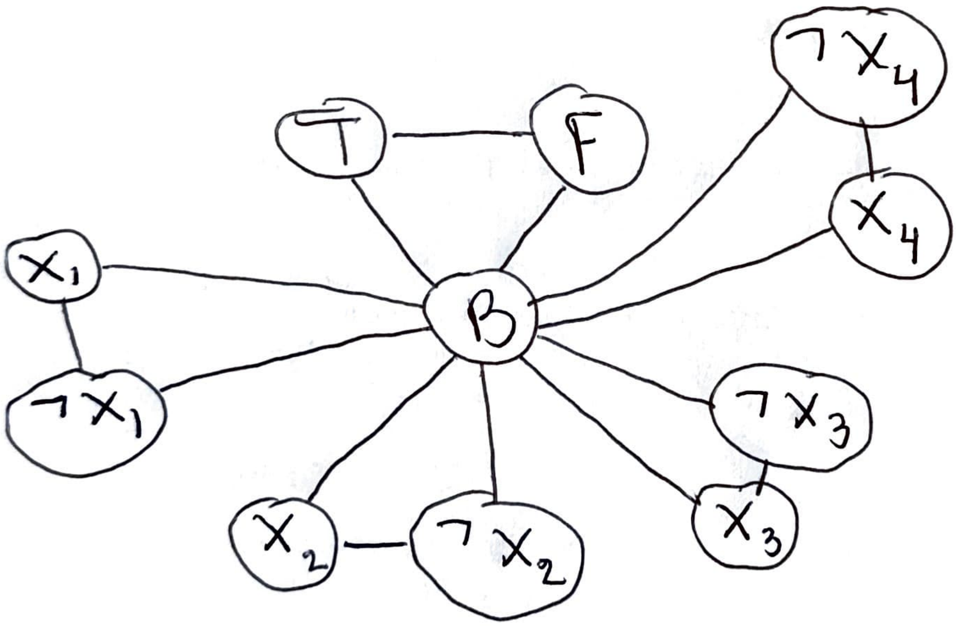
Step 1: 3 special vertices T, F, B C_4

for $i = 1, 2, \ldots, n$: 2 vertices x_i, $\neg x_i$

edges



if $n = 4$:



Property 1: Consider arbitrary 3-coloring. C5

We may assume: T is green

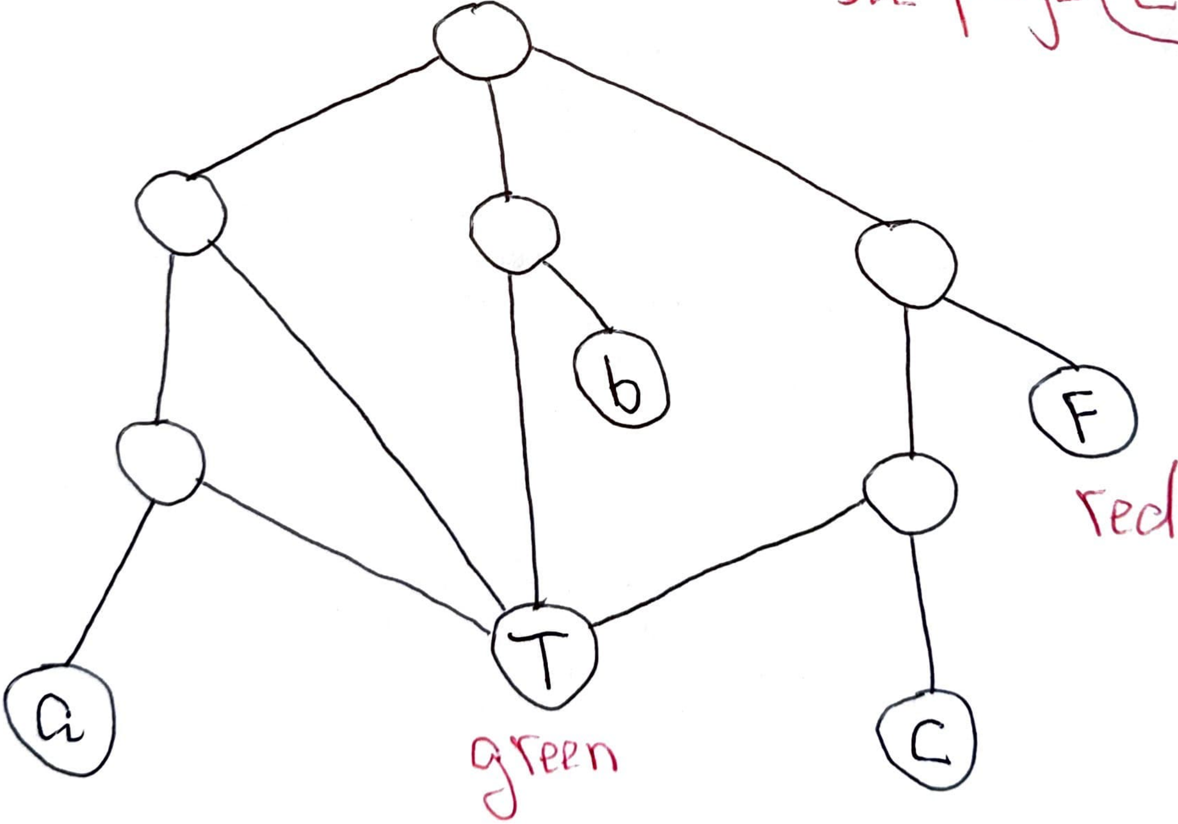
F is red

B is blue

for $i = 1, \dots, n$: one of x_i and $\neg x_i$ is green, the other is red.

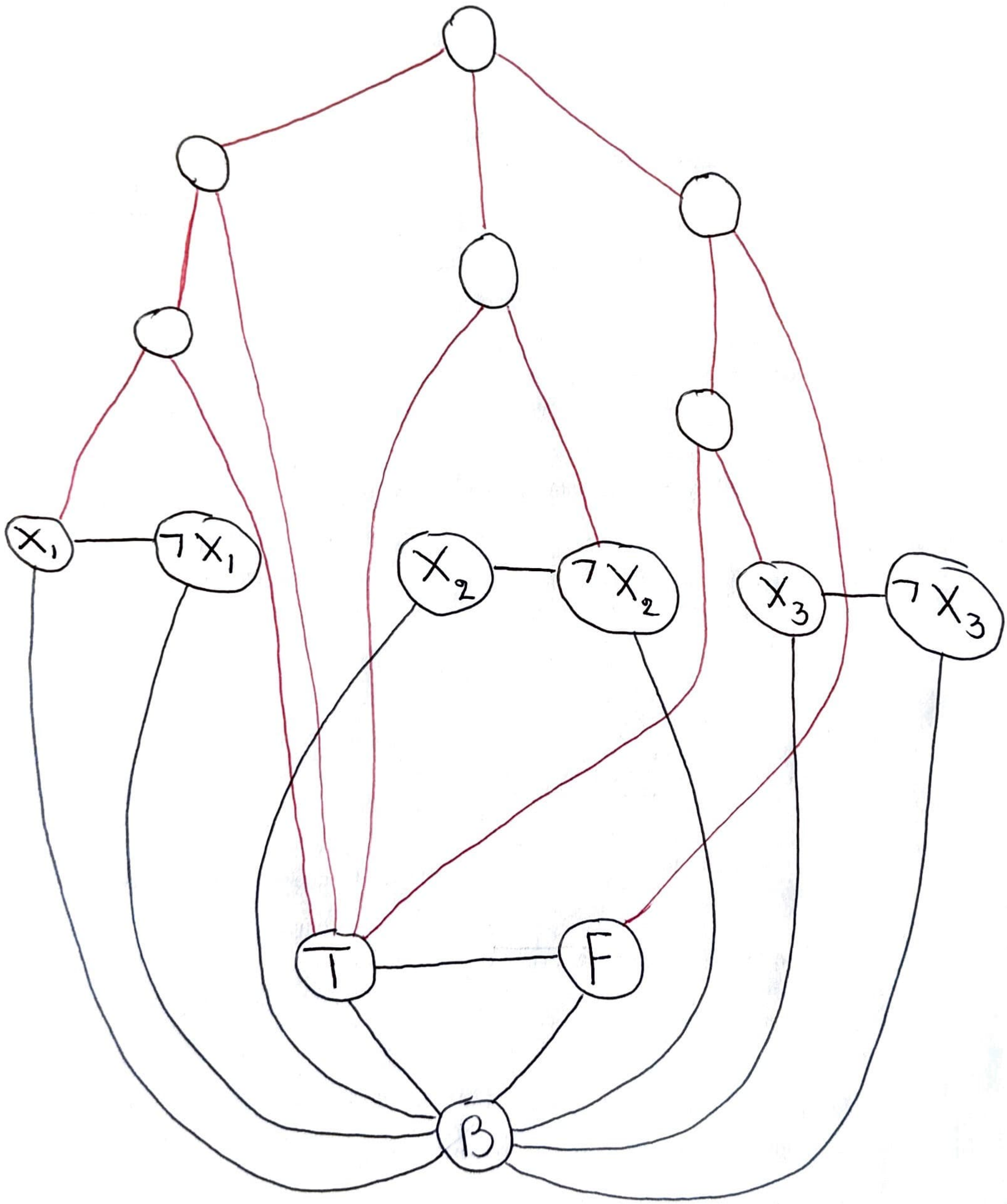
Step 2: for $j = 1, 2, \dots, m$: clause $C_j = a \vee b \vee c$

See page C5.5



Clause $x_1 \vee \neg x_2 \vee x_3$

C 5.5

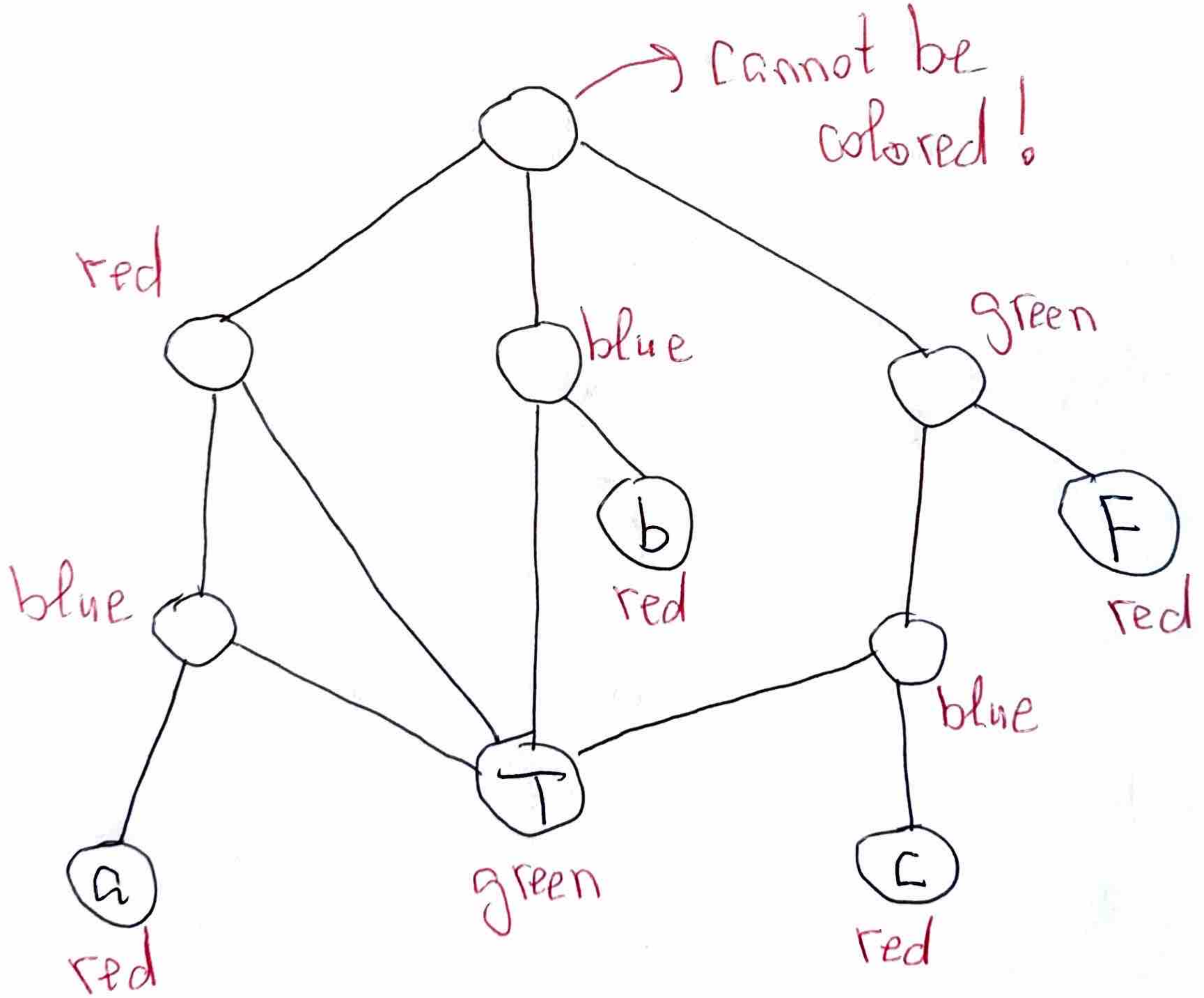


C6

<u>Property 2</u>: in any 3-coloring:

at least one of a, b, c is green.

<u>Proof</u>: From Property 1: each a, b, c is red or green. Assume they are all red.



□

Property 3: if at least one a, b, c (C7) is green, then the subgraph on page (C5) can be 3-colored.

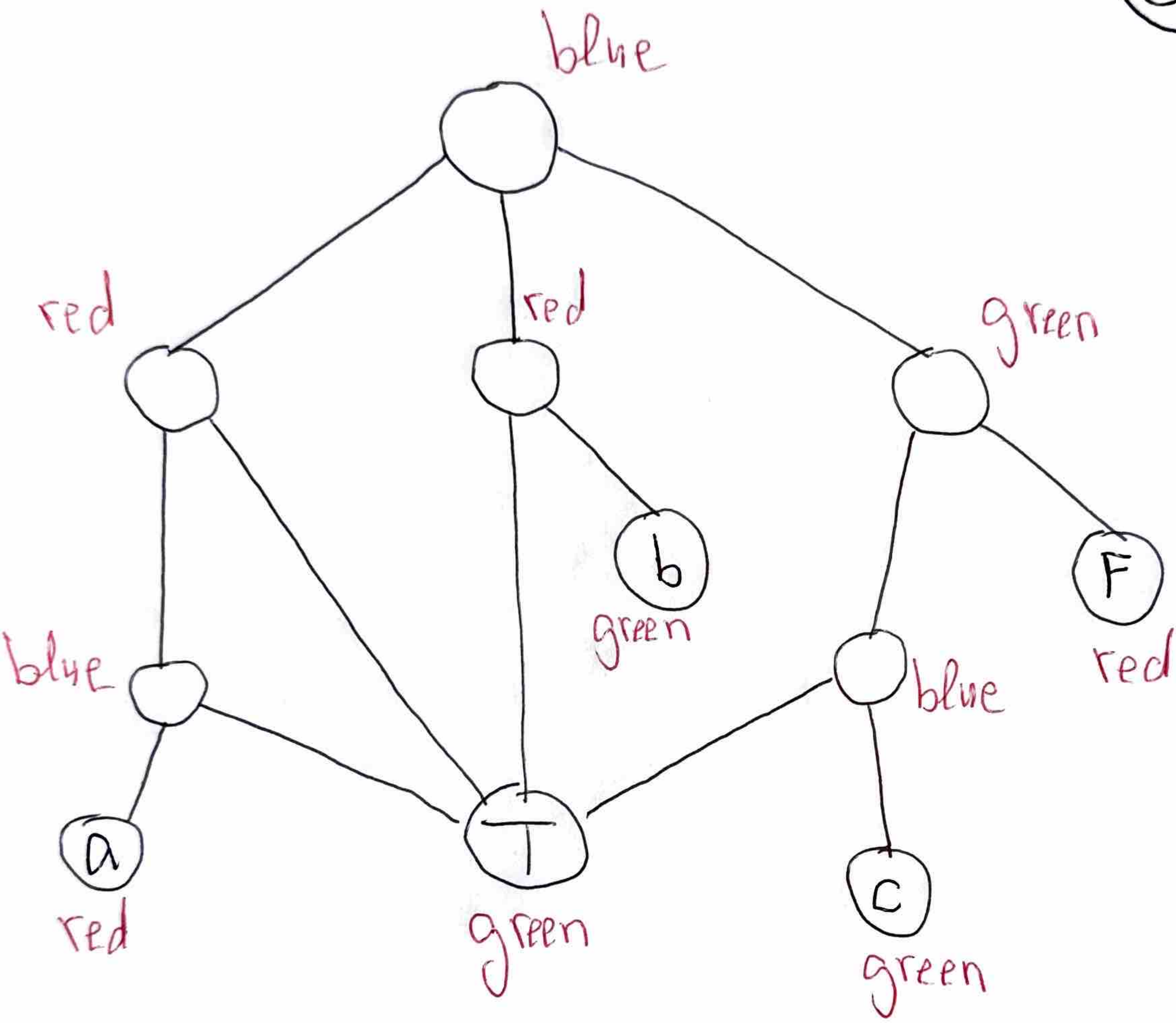
Proof: 7 cases:

a	b	c
g	g	g
g	g	r
g	r	g
r	g	g
g	r	r
r	g	r
r	r	g

see next page ← (r g g)

Exercise: verify the other 6 cases.

C8



$\square$

Steps 1 and 2 map φ to graph G C9

number of vertices: $3 + 2n + 6m$

number of edges: $3 + 3n + 13m$

Time to construct G: $O(n+m) = O(|\varphi|)$

Assume φ is satisfiable.

To show: G is 3-colorable.

truth values for $x_1, \dots, x_n$ such that $\varphi = \text{true}$.

3 - coloring of G: C_{10}

T green, F red, B blue

for $i = 1, \ldots, n$:

if x_i = true: x_i green, $\neg x_i$ red

if x_i = false: x_i red, $\neg x_i$ green

For each $j = 1, \ldots, m$: clause $C_j = a \vee b \vee c$
$= \text{true}$

at least one literal in C_j is true.

Let's say a = true (same reasoning if b = true or c = true)

if $a = x_i$: x_i green

if $a = \neg x_i$: $\neg x_i$ green

$\therefore$ in the subgraph corresponding to C_j, at least one of a, b, c is green.

$\therefore$ from Property 3: this subgraph C_{11} can be 3-colored.

Conclusion: the entire graph G can be 3-colored.

Assume G is 3-colorable.

To show: φ is satisfiable.

From Property 1:

T green, F red, B blue

for $i = 1, \ldots, n$: one of x_i and $\neg x_i$ is green, the other is red.

truth values for x_i, $i = 1, \dots, n$: C_{12}

if x_i green : set $x_i = \text{true}$

if $\neg x_i$ green: set $x_i = \text{false}$

This gives truth values for $x_1, \dots, x_n$

To show : $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_m = \text{true}$.

Let $j \in \{1, 2, \dots, m\}$.

To show: $C_j = \text{true}$.

$C_j = a \vee b \vee c$.

Assume $C_j = \text{false}$. Then $a = b = c = \text{false}$

if $a = x_i$: $\neg x_i$ green $\therefore$ x_i red

if $a = \neg x_i$: x_i green $\therefore$ $\neg x_i$ red

$\therefore$ in the subgraph corresponding to $C_j = a \vee b \vee c$, the three vertices a, b, c are red.

From Property 2: at least one of them is green.

$\lightning$

$\therefore$ $C_j =$ true.

Conclusion: φ is satisfiable.

Overall conclusion:

$$\left.\begin{array}{l} \text{3SAT is } NP\text{-complete} \\ \text{3Color is in } NP \\ \text{3SAT} \le_P \text{3Color} \end{array}\right\} \therefore \text{3Color is } NP\text{-complete.}$$